

Mismatched Multi-letter Successive Decoding for the Multiple-Access Channel

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Abstract

This paper studies channel coding for the discrete memoryless multiple-access channel with a given (possibly suboptimal) decoding rule. A multi-letter successive decoding rule depending on an arbitrary non-negative decoding metric is considered, and achievable rate regions and error exponents are derived both for the standard MAC (independent codebooks), and for the cognitive MAC (one user knows both messages) with superposition coding. In the cognitive case, the rate region and error exponent are shown to be tight with respect to the ensemble average. The rate regions are compared with those of the maximum-metric decoder, and numerical examples are given for which successive decoding yields a strictly higher sum rate for a given pair of input distributions.

I. INTRODUCTION

The mismatched decoding problem [1]–[3] seeks to characterize the performance of channel coding when the decoding rule is fixed and possibly suboptimal (e.g., due to channel uncertainty or implementation constraints). Extensions of this problem to multiuser settings are not only of interest in their own right, but can also provide valuable insight into the single-user setting [3]–[5]. In particular, significant attention has been paid to the mismatched multiple-access channel (MAC), described as follows. User $\nu = 1, 2$ transmits a codeword \mathbf{x}_ν from a codebook $\mathcal{C}_\nu = \{\mathbf{x}_\nu^{(1)}, \dots, \mathbf{x}_\nu^{(M_\nu)}\}$, and the output sequence \mathbf{y} is generated according to $W^n(\mathbf{y}|\mathbf{x}_1, \mathbf{x}_2) \triangleq \prod_{i=1}^n W(y_i|x_{1,i}, x_{2,i})$ for some transition law $W(y|x_1, x_2)$. The *mismatched* decoder estimates the message pair as

$$(\hat{m}_1, \hat{m}_2) = \arg \max_{(i,j)} q^n(\mathbf{x}_1^{(i)}, \mathbf{x}_2^{(j)}, \mathbf{y}), \quad (1)$$

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where $q^n(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}) \triangleq \prod_{i=1}^n q(x_{1,i}, x_{2,i}, y_i)$ for some non-negative decoding metric $q(x_1, x_2, y)$. The case $q(x_1, x_2, y) = W(y|x_1, x_2)$ corresponds to optimal maximum-likelihood (ML) decoding, whereas the introduction of mismatch can significantly increase the error probability and lead to smaller achievable rate regions. Even in the single-user case, characterizing the capacity with mismatch is a long-standing open problem.

Given that the decoder only knows the metric $q^n(\mathbf{x}_1^{(i)}, \mathbf{x}_2^{(j)}, \mathbf{y})$ corresponding to each codeword pair, one may question whether there exists a decoding rule with better performance than the maximum-metric rule in (1). In general, this question is only interesting if attention is restricted to “reasonable” decoding rules. For example, if the values $\{\log q(x_1, x_2, y)\}$ are rationally independent (i.e., no values can be written as linear combinations of the others with rational coefficients), then there is a one-to-one correspondence between the joint empirical distribution of $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y})$ and the possible values of $q^n(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y})$, and hence the decoder can implement the maximum-likelihood (ML) rule.

Nevertheless, there are a variety of possible decoding rules that are of interest beyond maximum-metric, including threshold decoding [6], [7], likelihood decoding [8], [9], and successive decoding [10], [11]. The latter has shown to be useful in a variety of multiple-access scenarios, and is the focus of the present paper. Specifically, we study the following two-step decoding rule:

$$\hat{\mathbf{m}}_1 = \arg \max_i \sum_j q^n(\mathbf{x}_1^{(i)}, \mathbf{x}_2^{(j)}, \mathbf{y}) \quad (2)$$

$$\hat{\mathbf{m}}_2 = \arg \max_j q^n(\mathbf{x}_1^{(\hat{\mathbf{m}}_1)}, \mathbf{x}_2^{(j)}, \mathbf{y}). \quad (3)$$

The rule in (2) is *multi-letter*, in the sense that the objective function does not factorize into a product of n symbols on $(\mathcal{X}_1, \mathcal{Y})$. Single-letter successive decoders [10, Sec. 4.5.1] could also potentially be studied from a mismatched decoding perspective by introducing a second decoding metric $q_2(x_1, y)$, but we focus on the above rule depending only on a *single* metric $q(x_1, x_2, y)$.

Under the above definitions of W , q , W^n and q^n , and assuming the corresponding alphabets \mathcal{X}_1 , \mathcal{X}_2 and \mathcal{Y} to be finite, we consider two distinct classes of MACs:

- 1) For the *standard MAC* [3], encoder $\nu = 1, 2$ takes as input \mathbf{m}_ν equiprobable on $\{1, \dots, M_\nu\}$, and transmits the corresponding codeword $\mathbf{x}_\nu^{(\mathbf{m}_\nu)}$ from a codebook \mathcal{C}_ν .
- 2) For the *cognitive MAC* [4] (or *MAC with degraded message sets* [10, Ex. 5.18]), the messages \mathbf{m}_ν are still equiprobable on $\{1, \dots, M_\nu\}$, but user 2 has access to both messages, while user 1 only knows \mathbf{m}_1 . Thus, \mathcal{C}_1 contains codewords indexed as $\mathbf{x}_1^{(i)}$, and \mathcal{C}_2 contains codewords indexed as $\mathbf{x}_2^{(i,j)}$.

For each of these, we say that a rate pair (R_1, R_2) is achievable if, for all $\delta > 0$, there exist sequences of codebooks $\mathcal{C}_{1,n}$ and $\mathcal{C}_{2,n}$ with $M_1 \geq e^{n(R_1 - \delta)}$ and $M_2 \geq e^{n(R_2 - \delta)}$ respectively, such that the error probability

$$p_e \triangleq \mathbb{P}[(\hat{\mathbf{m}}_1, \hat{\mathbf{m}}_2) \neq (\mathbf{m}_1, \mathbf{m}_2)] \quad (4)$$

tends to zero under the decoding rule described by (2)–(3). Our results will not depend on the method for breaking ties, so for concreteness, we assume that ties are broken as errors.

For fixed rates R_1 and R_2 , an error exponent $E(R_1, R_2)$ is said to be achievable if there exists a sequence of codebooks $\mathcal{C}_{1,n}$ and $\mathcal{C}_{2,n}$ with $M_1 \geq \exp(nR_1)$ and $M_2 \geq \exp(nR_2)$ codewords of length n such that

$$\liminf_{n \rightarrow \infty} -\frac{1}{n} \log p_e \geq E(R_1, R_2). \quad (5)$$

Letting $\mathcal{E}_\nu \triangleq \{\hat{m}_\nu \neq m_\nu\}$ for $\nu = 1, 2$, we observe that if $q(x_1, x_2, y) = W(y|x_1, x_2)$, then (2) is the decision rule that minimizes $\mathbb{P}[\mathcal{E}_1]$. Using this observation, we show in Appendix A that the successive decoder with $q = W$ is guaranteed to achieve the same rate region and error exponent as that of optimal non-successive maximum-likelihood decoding.

Since (2) minimizes $\mathbb{P}[\mathcal{E}_1]$, this step can be considered a mismatched version of the optimal decoding rule for (one user of) the interference channel. Thus, as well as giving an achievable rate region for the MAC with mismatched successive decoding ((2)–(3)), our results can be used to quantify the loss due to mismatch for the interference channel. In particular, we provide an achievable error exponent which is obtained using different techniques to those of [12].

A. Previous Work and Contributions

The vast majority of previous works on mismatched decoding have focused on achievability results via random coding, and the only general converse results are written in terms of non-computable information-spectrum type quantities [7]. For the point-to-point setting with mismatch, the asymptotics of random codes with independent codewords are well-understood for the i.i.d. [13], constant-composition [1], [14]–[16] and cost-constrained [2], [17] ensembles. Dual expressions and continuous alphabets were studied in [16] and [2].

The mismatched MAC was introduced by Lapidoth [3], who showed that (R_1, R_2) is achievable provided that

$$R_1 \leq \min_{\substack{\tilde{P}_{X_1 X_2 Y} : \tilde{P}_{X_1} = Q_1, \tilde{P}_{X_2 Y} = P_{X_2 Y}, \\ \mathbb{E}_{\tilde{P}}[\log q(X_1, X_2, Y)] \geq \mathbb{E}_P[\log q(X_1, X_2, Y)]}} I_{\tilde{P}}(X_1; X_2, Y) \quad (6)$$

$$R_2 \leq \min_{\substack{\tilde{P}_{X_1 X_2 Y} : \tilde{P}_{X_2} = Q_2, \tilde{P}_{X_1 Y} = P_{X_1 Y}, \\ \mathbb{E}_{\tilde{P}}[\log q(X_1, X_2, Y)] \geq \mathbb{E}_P[\log q(X_1, X_2, Y)]}} I_{\tilde{P}}(X_2; X_1, Y) \quad (7)$$

$$R_1 + R_2 \leq \min_{\substack{\tilde{P}_{X_1 X_2 Y} : \tilde{P}_{X_1} = Q_1, \tilde{P}_{X_2} = Q_2, \tilde{P}_Y = P_Y \\ \mathbb{E}_{\tilde{P}}[\log q(X_1, X_2, Y)] \geq \mathbb{E}_P[\log q(X_1, X_2, Y)] \\ I_{\tilde{P}}(X_1; Y) \leq R_1, I_{\tilde{P}}(X_2; Y) \leq R_2}} D(\tilde{P}_{X_1 X_2 Y} \| Q_1 \times Q_2 \times \tilde{P}_Y), \quad (8)$$

where Q_1 and Q_2 are arbitrary input distributions, and $P_{X_1 X_2 Y} \triangleq Q_1 \times Q_2 \times W$. The corresponding ensemble-tight error exponent was given by the present authors in [5], along with equivalent dual expressions and generalizations to continuous alphabets.

The mismatched cognitive MAC was introduced by Somekh-Baruch [4], who used superposition coding to show

that (R_1, R_2) is achievable provided that

$$R_2 \leq \min_{\substack{\tilde{P}_{X_1 X_2 Y} : \tilde{P}_{X_1 X_2} = Q_{X_1 X_2}, \tilde{P}_{X_1 Y} = P_{X_1 Y}, \\ \mathbb{E}_{\tilde{P}}[\log q(X_1, X_2, Y)] \geq \mathbb{E}_P[\log q(X_1, X_2, Y)]}} I_{\tilde{P}}(X_2; Y | X_1) \quad (9)$$

$$R_1 + R_2 \leq \min_{\substack{\tilde{P}_{X_1 X_2 Y} : \tilde{P}_{X_1 X_2} = Q_{X_1 X_2}, \tilde{P}_Y = P_Y, \\ \mathbb{E}_{\tilde{P}}[\log q(X_1, X_2, Y)] \geq \mathbb{E}_P[\log q(X_1, X_2, Y)], I_{\tilde{P}}(X_1, :, Y) \leq R_1}} I_{\tilde{P}}(X_1, X_2; Y), \quad (10)$$

where $Q_{X_1 X_2}$ is an arbitrary input distributions, and $P_{X_1 X_2 Y} \triangleq Q_{X_1 X_2} \times W$. The corresponding ensemble-tight error exponent was also given therein. Various forms of superposition coding were also studied by the present authors in [5], but with a focus on the single-user channel as opposed to the cognitive MAC.

Both of the above regions are known to be tight with respect to the ensemble average for constant-composition random coding, meaning that any looseness is due to the random-coding ensemble itself, rather than the bounding techniques used in the analysis [3], [4]. This concept was first explored in the single-user setting in [16]. We also note that the above regions lead to improved achievability bounds for the single-user setting [3], [4].

The main contributions of this paper are achievable rate regions for both the standard MAC (Section II-A) and cognitive MAC (Section II-B) under the successive decoding rule in (2)–(3). For the cognitive case, we also provide an ensemble tightness result. Both regions are numerically compared to their counterparts for maximum-metric decoding, and in each case, it is observed that the successive rule can provide a strictly higher sum rate, though neither the successive nor maximum-metric region is included in the other in general.

A by-product of our analysis is achievable error exponents corresponding to the rate regions. Our exponent for the standard MAC is related to that of Etkin *et al.* [18] for the interference channel, as both use parallel coding. Similarly, our exponent for the cognitive MAC is related to that of Kaspi and Merhav [19], since both use superposition coding. Like these works, we make use of type class enumerators; however, a key difference is that we avoid applying a Gallager-type bound in the initial step, and we instead proceed immediately with type-based methods. Consequently, our exponents are written only in terms of minimizations over “primal” parameters (joint distributions), as opposed to a combination also containing maximizations over “dual” parameters. Among all of these exponents, our exponent for the cognitive MAC is the only one for which ensemble tightness is proved.

After posting the initial version of this work, the interference channel perspective was pursued in depth in the *matched* case in an independent work [20]. It was shown that even without mismatch, improvements on the best known interference channel error exponents are attained. Moreover, whereas our focus is solely on codebooks with independent codewords, error exponents were also given for the Han-Kobayashi construction in [20].

B. Notation

Bold symbols are used for vectors (e.g., \mathbf{x}), and the corresponding i -th entry is written using a subscript (e.g., x_i). Subscripts are used to denote the distributions corresponding to expectations and mutual informations (e.g., $\mathbb{E}_P[\cdot]$, $I_P(X; Y)$). The marginals of a joint distribution P_{XY} are denoted by P_X and P_Y . We write $P_X = \tilde{P}_X$ to denote element-wise equality between two probability distributions on the same alphabet. The set of all sequences of length

n with a given empirical distribution P_X (i.e., type [21, Ch. 2]) is denoted by $T^n(P_X)$, and similarly for joint types. We write $f(n) \doteq g(n)$ if $\lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{f(n)}{g(n)} = 0$, and similarly for $\dot{\leq}$ and $\dot{\geq}$. We write $[\alpha]^+ = \max(0, \alpha)$, and denote the indicator function by $\mathbb{1}\{\cdot\}$

II. MAIN RESULTS

A. Standard MAC

Before presenting our main result for the standard MAC, we state the random-coding distribution that is used in its proof. For $\nu = 1, 2$, we fix an input distribution $Q_\nu \in \mathcal{P}(\mathcal{X}_\nu)$, and let $Q_{\nu,n}$ be a type with the same support as Q_ν such that $\max_{x_\nu} |Q_{\nu,n}(x_\nu) - Q_\nu(x_\nu)| \leq \frac{1}{n}$. We set

$$P_{\mathbf{X}_\nu}(\mathbf{x}_\nu) = \frac{1}{|T^n(Q_{\nu,n})|} \mathbb{1}\{\mathbf{x}_\nu \in T^n(Q_{\nu,n})\}, \quad (11)$$

and consider codewords $\{\mathbf{X}_\nu^{(i)}\}_{i=1}^{M_\nu}$ that are independently distributed according to $P_{\mathbf{X}_\nu}$. Thus,

$$\left(\{\mathbf{X}_1^{(i)}\}_{i=1}^{M_1}, \{\mathbf{X}_2^{(j)}\}_{j=1}^{M_2} \right) \sim \prod_{i=1}^{M_1} P_{\mathbf{X}_1}(\mathbf{x}_1^{(i)}) \prod_{j=1}^{M_2} P_{\mathbf{X}_2}(\mathbf{x}_2^{(j)}). \quad (12)$$

Our achievable rate region is written in terms of the functions

$$\begin{aligned} \overline{F}(\tilde{P}_{X_1 X_2 Y}, \tilde{P}'_{X_1 X_2 Y}, R_2) &\triangleq \max \left\{ \mathbb{E}_{\tilde{P}}[\log q(X_1, X_2, Y)], \right. \\ &\quad \left. \mathbb{E}_{\tilde{P}'}[\log q(X_1, X_2, Y)] + [R_2 - I_{\tilde{P}'}(X_2; X_1, Y)]^+ \right\} \end{aligned} \quad (13)$$

$$\begin{aligned} \underline{F}(P_{X_1 X_2 Y}, R_2) &\triangleq \max \left\{ \mathbb{E}_P[\log q(X_1, X_2, Y)], \right. \\ &\quad \left. \max_{P'_{X_1 X_2 Y} \in \mathcal{T}'_1(P_{X_1 X_2 Y}, R_2)} \mathbb{E}_{P'}[\log q(X_1, X_2, Y)] + R_2 - I_{P'}(X_2; X_1, Y) \right\}, \end{aligned} \quad (14)$$

and the sets

$$\begin{aligned} \mathcal{T}_1(P_{X_1 X_2 Y}, R_2) &\triangleq \left\{ (\tilde{P}_{X_1 X_2 Y}, \tilde{P}'_{X_1 X_2 Y}) : \tilde{P}_{X_2 Y} = P_{X_2 Y}, \tilde{P}_{X_1} = P_{X_1}, \right. \\ &\quad \left. \tilde{P}'_{X_1 Y} = \tilde{P}_{X_1 Y}, \tilde{P}'_{X_2} = P_{X_2}, \overline{F}(\tilde{P}_{X_1 X_2 Y}, \tilde{P}'_{X_1 X_2 Y}, R_2) \geq \underline{F}(P_{X_1 X_2 Y}, R_2) \right\} \end{aligned} \quad (15)$$

$$\mathcal{T}'_1(P_{X_1 X_2 Y}, R_2) \triangleq \left\{ P'_{X_1 X_2 Y} : P'_{X_1 Y} = P_{X_1 Y}, P'_{X_2} = P_{X_2}, I_{P'}(X_2; X_1, Y) \leq R_2 \right\} \quad (16)$$

$$\begin{aligned} \mathcal{T}_2(P_{X_1 X_2 Y}) &\triangleq \left\{ \tilde{P}_{X_1 X_2 Y} : \tilde{P}_{X_2} = P_{X_2}, \tilde{P}_{X_1 Y} = P_{X_1 Y}, \right. \\ &\quad \left. \mathbb{E}_{\tilde{P}}[\log q(X_1, X_2, Y)] \geq \mathbb{E}_P[\log q(X_1, X_2, Y)] \right\}. \end{aligned} \quad (17)$$

We will see in our analysis that $P_{X_1 X_2 Y}$ corresponds to the joint type of the transmitted codewords and the output sequence, and $\tilde{P}_{X_1 X_2 Y}$ corresponds to the joint type of some incorrect codeword of user 1, the transmitted codeword of user 2, and the output sequence. Moreover, $P'_{X_1 X_2 Y}$ and $\tilde{P}'_{X_1 X_2 Y}$ similarly correspond to joint types, the difference being that the X_2 marginal is instead associated with exponentially many sequences in the summation in (2).

Theorem 1. For any input distributions Q_1 and Q_2 , the pair (R_1, R_2) is achievable for the standard MAC with the mismatched successive decoding rule in (2)–(3) provided that

$$R_1 \leq \min_{(\tilde{P}_{X_1 X_2 Y}, \tilde{P}'_{X_1 X_2 Y}) \in \mathcal{T}_1(Q_1 \times Q_2 \times W, R_2)} I_{\tilde{P}}(X_1; X_2, Y) + [I_{\tilde{P}'}(X_2; X_1, Y) - R_2]^+ \quad (18)$$

$$R_2 \leq \min_{\tilde{P}_{X_1 X_2 Y} \in \mathcal{T}_2(Q_1 \times Q_2 \times W)} I_{\tilde{P}}(X_2; X_1, Y). \quad (19)$$

Proof: See Section III. ■

The minimization in (18) is a non-convex optimization problem, but it can be cast in terms of convex optimization problems, thus facilitating its computation. Details are provided in Appendix B.

While our focus is on achievable rates, the proof of Theorem 1 also provides error exponents. The exponent corresponding to (19) is precisely that corresponding to the error event for user 2 with maximum-metric decoding in [5, Sec. III], and the exponent corresponding to (18) is given by

$$\min_{P_{X_1 X_2 Y} : P_{X_1} = Q_1, P_{X_2} = Q_2} D(P_{X_1 X_2 Y} \| Q_1 \times Q_2 \times W) + [\tilde{I}_1(P_{X_1 X_2 Y}, R_2) - R_1]^+, \quad (20)$$

where $\tilde{I}_1(P_{X_1 X_2 Y}, R_2)$ denotes the right-hand side of (18) with an arbitrary distribution $P_{X_1 X_2 Y}$ in place of $Q_1 \times Q_2 \times W$.

Numerical Example: We consider the MAC with $\mathcal{X}_1 = \mathcal{X}_2 = \{0, 1\}$, $\mathcal{Y} = \{0, 1, 2\}$, and

$$W(y|x_1, x_2) = \begin{cases} 1 - 2\delta_{x_1 x_2} & y = x_1 + x_2 \\ \delta_{x_1 x_2} & \text{otherwise,} \end{cases} \quad (21)$$

where $\{\delta_{x_1 x_2}\}$ are constants. The mismatched decoder uses $q(x_1, x_2, y)$ of a similar form, but with a fixed value $\delta \in (0, \frac{1}{3})$ in place of $\{\delta_{x_1 x_2}\}$. While all such choices of δ are equivalent for maximum-metric decoding, this is not true for successive decoding.

We set $\delta_{00} = 0.01$, $\delta_{01} = 0.1$, $\delta_{10} = 0.01$, $\delta_{11} = 0.3$, $\delta = 0.15$, and $Q_1 = Q_2 = (0.5, 0.5)$. Figure 1 plots the achievable rates regions of successive decoding (Theorem 1), maximum-metric decoding ((6)–(8)), and matched decoding (yielding the same region whether successive or maximum-metric).

Interestingly, neither of the mismatched rate regions is included in the other, thus suggesting that the two decoding rules are fundamentally different. For the given input distribution, the sum rate for successive decoding exceeds that of maximum-metric decoding. Furthermore, upon taking the convex hull (which is justified by a time sharing argument), the region for successive decoding is strictly larger. While we observed similar behaviors for other choices of Q_1 and Q_2 , it remains unclear as to whether this is always the case. Furthermore, while the rate region for maximum-metric decoding is tight with respect to the ensemble average, it is unclear whether the same is true of the region given in Theorem 1.

The vertical line at $R_1 \approx 0.1$ is analogous to that in the interference channel, where for R_1 below a certain threshold, R_2 can be arbitrarily large while still ensuring that user 1's message is estimated correctly [12]. Due to the presence of mismatch, this induces a non-pentagonal shape in the present example. Note that the mismatched

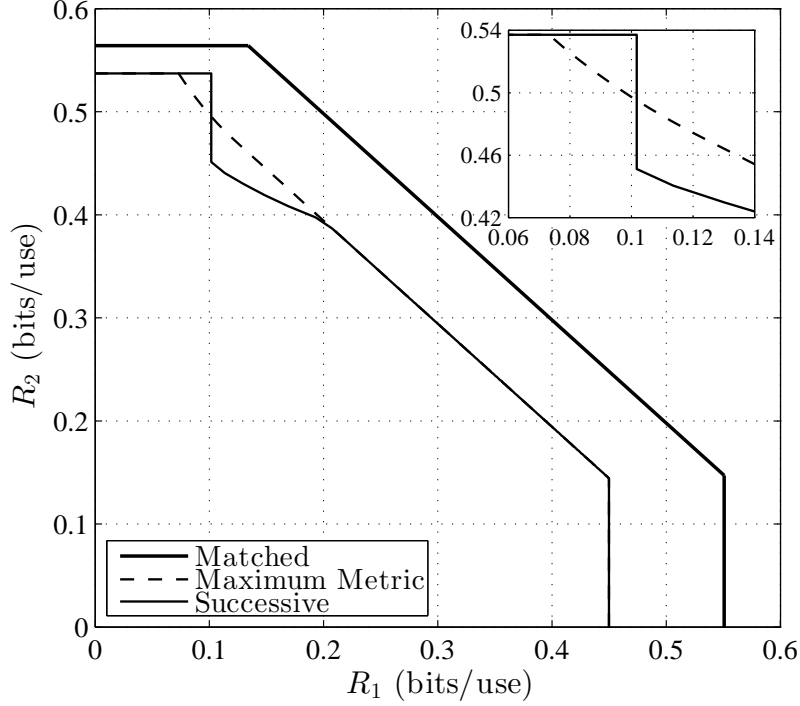


Figure 1. Achievable rate regions for the standard MAC given in (21) with mismatched successive decoding and mismatched maximum-metric decoding.

maximum-metric decoding region also has a non-pentagonal (and non-convex) shape, though its deviation from the usual pentagonal shape is milder than the successive decoder in this example.

B. Cognitive MAC

In this section, we consider the analog of Theorem 1 for the cognitive MAC. Besides being of interest in its own right, this will provide a case where ensemble-tightness can be established, and with the numerical results still exhibiting similar phenomena to those shown in Figure 1.

We again begin by introducing the random coding ensemble. We fix a joint distribution $Q_{X_1 X_2} \in \mathcal{P}(\mathcal{X}_1 \times \mathcal{X}_2)$, let $Q_{X_1 X_2, n}$ be the corresponding closest joint type in the same way as the previous subsection, and write the resulting marginals as Q_{X_1} , $Q_{X_1, n}$, $Q_{X_2|X_1}$, $Q_{X_2|X_1, n}$, and so on. We consider superposition coding, treating user 1's messages as the “cloud centers”, and user 2's messages as the “satellite codewords”. More precisely, defining

$$P_{\mathbf{X}_1}(\mathbf{x}_1) = \frac{1}{|T^n(Q_{X_1, n})|} \mathbb{1}\{\mathbf{x}_1 \in T^n(Q_{X_1, n})\} \quad (22)$$

$$P_{\mathbf{X}_2|\mathbf{X}_1}(\mathbf{x}_2|\mathbf{x}_1) = \frac{1}{|T_{\mathbf{x}_1}^n(Q_{X_2|X_1, n})|} \mathbb{1}\{\mathbf{x}_2 \in T_{\mathbf{x}_1}^n(Q_{X_2|X_1, n})\}, \quad (23)$$

the codewords are distributed as follows:

$$\left\{ \left(\mathbf{X}_1^{(i)}, \{\mathbf{X}_2^{(i,j)}\}_{j=1}^{M_2} \right) \right\}_{i=1}^{M_1} \sim \prod_{i=1}^{M_1} \left(P_{\mathbf{X}_1}(\mathbf{x}_1^{(i)}) \prod_{j=1}^{M_2} P_{\mathbf{X}_2|\mathbf{X}_1}(\mathbf{x}_2^{(i,j)}|\mathbf{x}_1^{(i)}) \right). \quad (24)$$

For the remaining definitions, we use similar notation to the standard MAC, with an additional subscript to avoid confusion. The analogous quantities to (13)–(17) are

$$\overline{F}_c(\tilde{P}'_{X_1 X_2 Y}, R_2) \triangleq \mathbb{E}_{\tilde{P}'}[\log q(X_1, X_2, Y)] + [R_2 - I_{\tilde{P}'}(X_2; Y|X_1)]^+ \quad (25)$$

$$\underline{F}_c(P_{X_1 X_2 Y}, R_2) \triangleq \max \left\{ \mathbb{E}_P[\log q(X_1, X_2, Y)], \right. \\ \left. \max_{P'_{X_1 X_2 Y} \in \mathcal{T}'_{1c}(P_{X_1 X_2 Y}, R_2)} \mathbb{E}_{P'}[\log q(X_1, X_2, Y)] + R_2 - I_{P'}(X_2; Y|X_1) \right\}, \quad (26)$$

$$\mathcal{T}_{1c}(P_{X_1 X_2 Y}, R_2) \triangleq \left\{ \tilde{P}'_{X_1 X_2 Y} : P'_{X_1 X_2} = P_{X_1 X_2}, \tilde{P}'_Y = P_Y, \overline{F}_c(\tilde{P}'_{X_1 X_2 Y}, R_2) \geq \underline{F}_c(P_{X_1 X_2 Y}, R_2) \right\} \quad (27)$$

$$\mathcal{T}'_{1c}(P_{X_1 X_2 Y}, R_2) \triangleq \left\{ P'_{X_1 X_2 Y} : P'_{X_1 Y} = P_{X_1 Y}, P'_{X_1 X_2} = P_{X_1 X_2}, I_{P'}(X_2; Y|X_1) \leq R_2 \right\} \quad (28)$$

$$\mathcal{T}_{2c}(P_{X_1 X_2 Y}) \triangleq \left\{ \tilde{P}_{X_1 X_2 Y} : \tilde{P}_{X_1 X_2} = P_{X_1 X_2}, \tilde{P}_{X_1 Y} = P_{X_1 Y}, \right. \\ \left. \mathbb{E}_{\tilde{P}}[\log q(X_1, X_2, Y)] \geq \mathbb{E}_P[\log q(X_1, X_2, Y)] \right\}. \quad (29)$$

Our main result for the cognitive MAC is as follows.

Theorem 2. *For any input distribution $Q_{X_1 X_2}$, the pair (R_1, R_2) is achievable for the cognitive MAC with the mismatched successive decoding rule in (2)–(3) provided that*

$$R_1 \leq \min_{\tilde{P}'_{X_1 X_2 Y} \in \mathcal{T}'_{1c}(Q_{X_1 X_2} \times W, R_2)} I_{\tilde{P}'}(X_1; Y) + [I_{\tilde{P}'}(X_2; Y|X_1) - R_2]^+ \quad (30)$$

$$R_2 \leq \min_{\tilde{P}_{X_1 X_2 Y} \in \mathcal{T}_{2c}(Q_{X_1 X_2} \times W)} I_{\tilde{P}}(X_2; Y|X_1). \quad (31)$$

Conversely, for any rate pair (R_1, R_2) failing to meet both of these conditions, the random-coding error probability resulting from (22)–(24) tends to one as $n \rightarrow \infty$.

Proof: See Section IV. ■

In Appendix B, we cast (30) in terms of convex optimization problems. Similarly to the previous subsection, the exponent corresponding to (31) is precisely that corresponding to the second user in [4, Thm. 1], and the exponent corresponding to (30) is given by

$$\min_{P_{X_1 X_2 Y} : P_{X_1 X_2} = Q_{X_1 X_2}} D(P_{X_1 X_2 Y} \| Q_{X_1 X_2} \times W) + [I_{0c}(P_{X_1 X_2 Y}, R_2) - R_1]^+, \quad (32)$$

where $I_{0c}(P_{X_1 X_2 Y}, R_2)$ denotes the right-hand side of (30) with an arbitrary distribution $P_{X_1 X_2 Y}$ in place of $Q_{X_1 X_2} \times W$. Similarly to the rate region, the proof of Theorem 2 shows that these exponents are tight with respect to the ensemble average (sometimes referred to as *exact random-coding exponents* [22]).

Numerical Example: We consider again consider the transition law (and the corresponding decoding metric with a single value of δ) given in (21) with $\delta_{00} = 0.01$, $\delta_{01} = 0.1$, $\delta_{10} = 0.01$, $\delta_{11} = 0.3$, $\delta = 0.15$, and $Q_{X_1 X_2} = Q_1 \times Q_2$ with $Q_1 = Q_2 = (0.5, 0.5)$. Figure 2 plots the achievable rates regions of successive decoding (Theorem 2), maximum-metric decoding ((9)–(10)), and matched decoding (again yielding the same region whether successive or maximum-metric, cf. Appendix A).

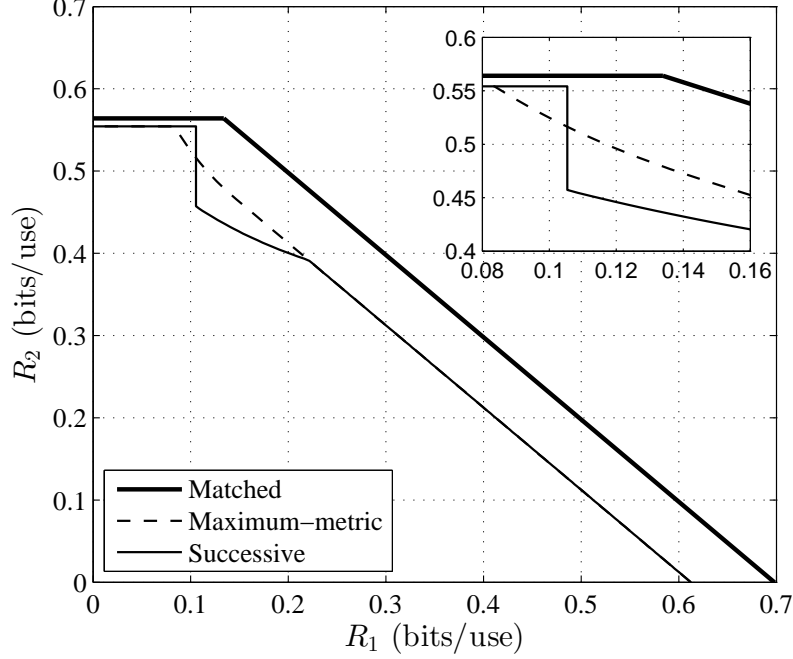


Figure 2. Achievable rate regions for the cognitive MAC given in (21) with mismatched successive decoding and mismatched maximum-metric decoding.

We see that the behavior of the decoders is completely analogous to the non-cognitive case observed in Figure 1. The key difference here is that we know that all three regions are tight with respect to the ensemble average. Thus, we may conclude that the somewhat unusual shape of the region for successive decoding is not merely an artifact of our analysis, but it is indeed inherent to the random-coding ensemble and the decoder. On the other hand, the “non-pentagonal” behavior for high values of R_1 is to be expected in the cognitive case, due to user 2 knowing the message of user 1 [10, Ex. 5.18].

III. PROOF OF THEOREM 1

Here we present the proof of Theorem 1. Our analysis is based on the method of type class enumeration (e.g. see [12], [22], [23]), and is perhaps most similar to that of Somekh-Baruch and Merhav [22].

Step 1: Initial Bound

We assume without loss of generality that $m_1 = m_2 = 1$, and we write $\mathbf{X}_\nu = \mathbf{X}_\nu^{(1)}$ and let $\bar{\mathbf{X}}_\nu$ denote an arbitrary codeword $\mathbf{X}_\nu^{(j)}$ with $j \neq 1$. Thus,

$$(\mathbf{X}_1, \mathbf{X}_2, \mathbf{Y}, \bar{\mathbf{X}}_1, \bar{\mathbf{X}}_2) \sim P_{\mathbf{X}_1}(\mathbf{x}_1)P_{\mathbf{X}_2}(\mathbf{x}_2)W^n(\mathbf{y}|\mathbf{x}_1, \mathbf{x}_2)P_{\mathbf{X}_1}(\bar{\mathbf{x}}_1)P_{\mathbf{X}_2}(\bar{\mathbf{x}}_2). \quad (33)$$

We define the following error events:

(Type 1) $\sum_j q^n(\mathbf{X}_1^{(i)}, \mathbf{X}_2^{(j)}, \mathbf{Y}) \geq \sum_j q^n(\mathbf{X}_1, \mathbf{X}_2^{(j)}, \mathbf{Y})$ for some $i \neq 1$;

(Type 2) $q^n(\mathbf{X}_1, \mathbf{X}_2^{(j)}, \mathbf{Y}) \geq q^n(\mathbf{X}_1, \mathbf{X}_2, \mathbf{Y})$ for some $j \neq 1$.

Denoting the probabilities of these events by $\bar{p}_{e,1}$ and $\bar{p}_{e,2}$ respectively, it follows that the overall random-coding error probability \bar{p}_e is upper bounded by $\bar{p}_{e,1} + \bar{p}_{e,2}$.

The analysis of the type-2 error event is precisely that of one of the three error types for maximum-metric decoding [3], [5], yielding the rate condition in (19). We thus focus on the type-1 event. We let $\bar{p}_{e,1}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y})$ denote the probability of the type-1 event conditioned on $(\mathbf{X}_1^{(1)}, \mathbf{X}_2^{(1)}, \mathbf{Y}) = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{y})$, and we denote the joint type of $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y})$ by $P_{X_1 X_2 Y}$. We write the objective function in (2) as

$$\Xi_{\mathbf{x}_2 \mathbf{y}}(\bar{\mathbf{x}}_1) \triangleq q^n(\bar{\mathbf{x}}_1, \mathbf{x}_2, \mathbf{y}) + \sum_{j \neq 1} q^n(\bar{\mathbf{x}}_1, \mathbf{X}_2^{(j)}, \mathbf{y}). \quad (34)$$

This quantity is random due to the randomness of $\{\mathbf{X}_2^{(j)}\}$. The starting point of our analysis is the union bound:

$$\bar{p}_{e,1}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}) \leq (M_1 - 1) \mathbb{P}[\Xi_{\mathbf{x}_2 \mathbf{y}}(\bar{\mathbf{X}}_1) \geq \Xi_{\mathbf{x}_2 \mathbf{y}}(\mathbf{x}_1)]. \quad (35)$$

The difficulty in analyzing (35) is that for two different codewords \mathbf{x}_1 and $\bar{\mathbf{x}}_1$, $\Xi_{\mathbf{x}_2 \mathbf{y}}(\mathbf{x}_1)$ and $\Xi_{\mathbf{x}_2 \mathbf{y}}(\bar{\mathbf{x}}_1)$ are not independent, and their joint statistics are complicated. We will circumvent this issue by conditioning on high probability events under which these random quantities can be bounded by deterministic values.

Step 2: An Auxiliary Lemma

We introduce some additional notation: For a given realization \mathbf{x}_1 of \mathbf{X}_1 , we let

$$N_{\bar{\mathbf{x}}_1 \mathbf{y}}(\tilde{P}'_{X_1 X_2 Y}) = \sum_{j \neq 1} \mathbb{1}\{(\bar{\mathbf{x}}_1, \mathbf{X}_2^{(j)}, \mathbf{y}) \in T^n(\tilde{P}'_{X_1 X_2 Y})\} \quad (36)$$

denote the random number of $\mathbf{X}_2^{(j)}$ ($j \neq 1$) such that $(\bar{\mathbf{x}}_1, \mathbf{X}_2^{(j)}, \mathbf{y}) \in T^n(\tilde{P}'_{X_1 X_2 Y})$, $\tilde{P}_{X_1 X_2 Y}$ is the joint type of $(\bar{\mathbf{x}}_1, \mathbf{x}_2, \mathbf{y})$, and we write $q^n(\tilde{P}'_{X_1 X_2 Y}) \triangleq q^n(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \mathbf{y})$ for an arbitrary triplet $(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \mathbf{y}) \in T^n(\tilde{P}_{X_1 X_2 Y})$.

The key to replacing random quantities by deterministic ones is to condition on events that hold with probability one approaching super-exponentially fast, thus not affecting the exponential behavior of interest. The following lemma will be used for this purpose, characterizing the behavior of $N_{\bar{\mathbf{x}}_1 \mathbf{y}}(\tilde{P}'_{X_1 X_2 Y})$ for various choices of R_2 and $\tilde{P}'_{X_1 X_2 Y}$. The proof can be found in [12], [22], and is based on the fact that

$$\mathbb{P}[(\bar{\mathbf{x}}_1, \bar{\mathbf{X}}_2, \mathbf{y}) \in T^n(\tilde{P}'_{X_1 X_2 Y})] \doteq e^{-nI_{\tilde{P}'}(X_2; X_1, Y)}, \quad (37)$$

which is a standard property of types [21, Ch. 2].

Lemma 1. [12], [22] *Fix the pair $(\bar{\mathbf{x}}_1, \mathbf{y}) \in T^n(\tilde{P}_{X_1 Y})$, a constant $\delta > 0$, and a type $\tilde{P}'_{X_1 X_2 Y} \in \mathcal{S}'_{1,n}(Q_{2,n}, \tilde{P}_{X_1 Y})$.*

1) If $R_2 \geq I_{\tilde{P}'}(X_2; X_1, Y) + \delta$, then

$$M_2 e^{-n(I_{\tilde{P}'}(X_2; X_1, Y) + \delta)} \leq N_{\bar{\mathbf{x}}_1 \mathbf{y}}(\tilde{P}'_{X_1 X_2 Y}) \leq M_2 e^{-n(I_{\tilde{P}'}(X_2; X_1, Y) - \delta)} \quad (38)$$

with probability approaching one super-exponentially fast.

2) If $R_2 < I_{\tilde{P}'}(X_2; X_1, Y) + \delta$, then

$$N_{\bar{\mathbf{x}}_1 \mathbf{y}}(\tilde{P}'_{X_1 X_2 Y}) \leq e^{n 2\delta} \quad (39)$$

with probability approaching one super-exponentially fast.

Roughly speaking, Lemma 1 states that if $R_2 > I_{\tilde{P}'}(X_2; X_1, Y)$ then the type enumerator is highly concentrated about its mean, whereas if $R_2 < I_{\tilde{P}'}(X_2; X_1, Y)$ then the type enumerator takes a subexponential value (possibly zero) with overwhelming probability.

Given a joint type $\tilde{P}_{X_1 Y}$, define the event

$$\begin{aligned} \mathcal{A}_\delta(\tilde{P}_{X_1 Y}) = & \left\{ (38) \text{ holds for all } \tilde{P}'_{X_1 X_2 Y} \in \mathcal{S}'_{1,n}(Q_2, \tilde{P}_{X_1 Y}) \text{ with } R_2 \geq I_{\tilde{P}'}(X_2; X_1, Y) + \delta \right\} \\ & \cap \left\{ (39) \text{ holds for all } \tilde{P}'_{X_1 X_2 Y} \in \mathcal{S}'_{1,n}(Q_2, \tilde{P}_{X_1 Y}) \text{ with } R_2 < I_{\tilde{P}'}(X_2; X_1, Y) + \delta \right\}. \end{aligned} \quad (40)$$

By Lemma 1 and the union bound, $\mathbb{P}[\mathcal{A}_\delta(\tilde{P}_{X_1 Y})] \rightarrow 1$ super-exponentially fast, and hence we can safely condition any event on $\mathcal{A}_\delta(\tilde{P}_{X_1 Y})$ without changing the exponential behavior of the corresponding probability. This can be seen by writing the following for any event \mathcal{E} :

$$\mathbb{P}[\mathcal{E}] = \mathbb{P}[\mathcal{E} \cap \mathcal{A}] + \mathbb{P}[\mathcal{E} \cap \mathcal{A}^c] \quad (41)$$

$$\leq \mathbb{P}[\mathcal{E} \mid \mathcal{A}] + \mathbb{P}[\mathcal{A}^c] \quad (42)$$

$$\mathbb{P}[\mathcal{E}] \geq \mathbb{P}[\mathcal{E} \cap \mathcal{A}] \quad (43)$$

$$= (1 - \mathbb{P}[\mathcal{A}^c])\mathbb{P}[\mathcal{E} \mid \mathcal{A}] \quad (44)$$

$$\geq \mathbb{P}[\mathcal{E} \mid \mathcal{A}] - \mathbb{P}[\mathcal{A}^c]. \quad (45)$$

Using these observations, we will condition on \mathcal{A}_δ several times throughout the remainder of the proof.

Step 3: Bound $\Xi_{x_2 \mathbf{y}}(x_1)$ by a Deterministic Quantity

From (34), we have

$$\Xi_{x_2 \mathbf{y}}(\bar{\mathbf{x}}_1) = q^n(\tilde{P}_{X_1 X_2 Y}) + \sum_{\tilde{P}'_{X_1 X_2 Y}} N_{\bar{\mathbf{x}}_1 \mathbf{y}}(\tilde{P}'_{X_1 X_2 Y}) q^n(\tilde{P}'_{X_1 X_2 Y}). \quad (46)$$

Since the codewords are generated independently, $N_{\bar{\mathbf{x}}_1 \mathbf{y}}(\tilde{P}'_{X_1 X_2 Y})$ is binomially distributed with $M_2 - 1$ trials and success probability $\mathbb{P}[(\bar{\mathbf{x}}_1, \bar{\mathbf{X}}_2, \mathbf{y}) \in T^n(\tilde{P}'_{X_1 X_2 Y})]$. By construction, we have $N_{\bar{\mathbf{x}}_1 \mathbf{y}}(\tilde{P}'_{X_1 X_2 Y}) = 0$ unless $\tilde{P}'_{X_1 X_2 Y} \in \mathcal{S}'_{1,n}(Q_{2,n}, \tilde{P}_{X_1 Y})$, where

$$\mathcal{S}'_{1,n}(Q_{2,n}, \tilde{P}_{X_1 Y}) \triangleq \left\{ \tilde{P}'_{X_1 X_2 Y} \in \mathcal{P}_n(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y}) : \tilde{P}'_{X_1 Y} = \tilde{P}_{X_1 Y}, \tilde{P}'_{X_2} = Q_{2,n} \right\}. \quad (47)$$

Conditioned on $\mathcal{A}_\delta(P_{X_1Y})$, we have the following:

$$\Xi_{\mathbf{x}_2\mathbf{y}}(\mathbf{x}_1) = q^n(P_{X_1X_2Y}) + \sum_{P'_{X_1X_2Y}} N_{\mathbf{x}_1\mathbf{y}}(P'_{X_1X_2Y}) q^n(P'_{X_1X_2Y}) \quad (48)$$

$$\geq q^n(P_{X_1X_2Y}) + \max_{\substack{P'_{X_1X_2Y} \in \mathcal{S}'_{1,n}(Q_{2,n}, P_{X_1Y}) \\ R_2 \geq I_{P'}(X_2; X_1, Y) + \delta}} N_{\mathbf{x}_1\mathbf{y}}(P'_{X_1X_2Y}) q^n(P'_{X_1X_2Y}) \quad (49)$$

$$\geq q^n(P_{X_1X_2Y}) + \max_{\substack{P'_{X_1X_2Y} \in \mathcal{S}'_{1,n}(Q_{2,n}, P_{X_1Y}) \\ R_2 \geq I_{P'}(X_2; X_1, Y) + \delta}} M_2 e^{-n(I_{P'}(X_2; X_1, Y) + \delta)} q^n(P'_{X_1X_2Y}) \quad (50)$$

$$\triangleq \underline{G}_{\delta,n}(P_{X_1X_2Y}), \quad (51)$$

where (50) follows from (38). Unlike $\Xi_{\mathbf{x}_2\mathbf{y}}(\mathbf{x}_1)$, the quantity $\underline{G}_{\delta,n}(P_{X_1X_2Y})$ is deterministic. Substituting (51) into (35) and using the fact that $\mathbb{P}[\mathcal{A}_\delta(\tilde{P}_{X_1Y})] \rightarrow 1$ super-exponentially fast, we obtain

$$\bar{p}_{e,1}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}) \leq M_1 \mathbb{P}[\Xi_{\mathbf{x}_2\mathbf{y}}(\bar{\mathbf{X}}_1) \geq \underline{G}_{\delta,n}(P_{X_1X_2Y})]. \quad (52)$$

Step 4: An Expansion Based on Types

Since the statistics of $\Xi_{\mathbf{x}_2\mathbf{y}}(\bar{\mathbf{x}}_1)$ depend on $\bar{\mathbf{x}}_1$ only through the joint type of $(\bar{\mathbf{x}}_1, \mathbf{x}_2, \mathbf{y})$, we can write (52) as follows:

$$\bar{p}_{e,1}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}) \leq M_1 \sum_{\tilde{P}_{X_1X_2Y}} \mathbb{P}[(\bar{\mathbf{X}}_1, \mathbf{x}_2, \mathbf{y}) \in T^n(\tilde{P}_{X_1X_2Y})] \mathbb{P}[\Xi_{\mathbf{x}_2\mathbf{y}}(\bar{\mathbf{x}}_1) \geq \underline{G}_{\delta,n}(P_{X_1X_2Y})] \quad (53)$$

$$\doteq M_1 \max_{\tilde{P}_{X_1X_2Y} \in \mathcal{S}_{1,n}(Q_{1,n}, P_{X_2Y})} e^{-nI_{\tilde{P}}(X_1; X_2, Y)} \mathbb{P}[\Xi_{\mathbf{x}_2\mathbf{y}}(\bar{\mathbf{x}}_1) \geq \underline{G}_{\delta,n}(P_{X_1X_2Y})], \quad (54)$$

where $\bar{\mathbf{x}}_1$ denotes an arbitrary sequence such that $(\bar{\mathbf{x}}_1, \mathbf{x}_2, \mathbf{y}) \in T^n(\tilde{P}_{X_1X_2Y})$, and

$$\mathcal{S}_{1,n}(Q_{1,n}, P_{X_2Y}) \triangleq \left\{ \tilde{P}_{X_1X_2Y} \in \mathcal{P}_n(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y}) : \tilde{P}_{X_1} = Q_{1,n}, \tilde{P}_{X_2Y} = P_{X_2Y} \right\}. \quad (55)$$

In (54), we have used an analogous property to (37) and the fact that by construction, the joint type of $(\bar{\mathbf{X}}_1, \mathbf{x}_2, \mathbf{y})$ is in $\mathcal{S}_{1,n}(Q_{1,n}, P_{X_2Y})$ with probability one.

Step 5: Bound $\Xi_{\mathbf{x}_2\mathbf{y}}(\bar{\mathbf{x}}_1)$ by a Deterministic Quantity

Next, we again use Lemma 1 in order to replace $\Xi_{\mathbf{x}_2\mathbf{y}}(\bar{\mathbf{x}}_1)$ in (54) by a deterministic quantity. We have from (46) that

$$\Xi_{\mathbf{x}_2\mathbf{y}}(\bar{\mathbf{x}}_1) \leq q^n(\tilde{P}_{X_1X_2Y}) + p_0(n) \max_{\tilde{P}'_{X_1X_2Y}} N_{\bar{\mathbf{x}}_1\mathbf{y}}(\tilde{P}'_{X_1X_2Y}) q^n(\tilde{P}'_{X_1X_2Y}), \quad (56)$$

where $p_0(n)$ is a polynomial corresponding to the total number of joint types. Substituting (56) into (54), we obtain

$$\bar{p}_{e,1}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}) \leq M_1 \max_{\tilde{P}_{X_1X_2Y} \in \mathcal{S}_{1,n}(Q_{1,n}, P_{X_2Y})} \max_{\tilde{P}'_{X_1X_2Y} \in \mathcal{S}'_1(Q_{2,n}, \tilde{P}_{X_1Y})} e^{-nI_{\tilde{P}}(X_1; X_2, Y)} \mathbb{P}[\mathcal{E}_{P, \tilde{P}}(\tilde{P}'_{X_1X_2Y})], \quad (57)$$

where

$$\mathcal{E}_{P, \tilde{P}}(\tilde{P}'_{X_1X_2Y}) \triangleq \left\{ q^n(\tilde{P}_{X_1X_2Y}) + p_0(n) N_{\bar{\mathbf{x}}_1\mathbf{y}}(\tilde{P}'_{X_1X_2Y}) q^n(\tilde{P}'_{X_1X_2Y}) \geq \underline{G}_{\delta,n}(P_{X_1X_2Y}) \right\}, \quad (58)$$

and we have used the union bound to take the maximum over $\tilde{P}'_{X_1 X_2 Y}$ outside the probability in (57). Continuing, we have for any $\tilde{P}_{X_1 X_2 Y} \in \mathcal{S}_{1,n}(Q_{1,n}, P_{X_2 Y})$ that

$$\max_{\tilde{P}'_{X_1 X_2 Y} \in \mathcal{S}'_{1,n}(Q_{2,n}, \tilde{P}_{X_1 Y})} \mathbb{P}[\mathcal{E}_{P, \tilde{P}}(\tilde{P}'_{X_1 X_2 Y})] = \max \left\{ \max_{\substack{\tilde{P}'_{X_1 X_2 Y} \in \mathcal{S}'_{1,n}(Q_{2,n}, \tilde{P}_{X_1 Y}) \\ R_2 \geq I_{\tilde{P}'}(X_2; X_1, Y) + \delta}} \mathbb{P}[\mathcal{E}_{P, \tilde{P}}(\tilde{P}'_{X_1 X_2 Y})], \right. \\ \left. \max_{\substack{\tilde{P}'_{X_1 X_2 Y} \in \mathcal{S}'_{1,n}(Q_{2,n}, \tilde{P}_{X_1 Y}) \\ R_2 < I_{\tilde{P}'}(X_2; X_1, Y) + \delta}} \mathbb{P}[\mathcal{E}_{P, \tilde{P}}(\tilde{P}'_{X_1 X_2 Y})] \right\}. \quad (59)$$

Step 5a – Simplify the First Maximization: For the first maximization in (59), observe that conditioned on $\mathcal{A}_\delta(\tilde{P}_{X_1 Y})$ in (40), we have for $\tilde{P}'_{X_1 X_2 Y}$ satisfying $R_2 \geq I_{\tilde{P}'}(X_2; X_1, Y) + \delta$ that

$$N_{\tilde{\mathbf{x}}_1 \mathbf{y}}(\tilde{P}'_{X_1 X_2 Y}) q^n(\tilde{P}'_{X_1 X_2 Y}) \leq M_2 e^{-n(I_{\tilde{P}'}(X_2; X_1, Y) - \delta)} q^n(\tilde{P}'_{X_1 X_2 Y}), \quad (60)$$

where we have used (38). Hence, and since $\mathbb{P}[\mathcal{A}_\delta(\tilde{P}_{X_1 Y})] \rightarrow 1$ super-exponentially fast, we have

$$\mathbb{P}[\mathcal{E}_{P, \tilde{P}}(\tilde{P}'_{X_1 X_2 Y})] \leq \mathbb{1} \left\{ q^n(\tilde{P}_{X_1 X_2 Y}) + M_2 p_0(n) e^{-n(I_{\tilde{P}'}(X_2; X_1, Y) - \delta)} q^n(\tilde{P}'_{X_1 X_2 Y}) \geq \underline{G}_{\delta, n}(P_{X_1 X_2 Y}) \right\}. \quad (61)$$

Step 5b – Simplify the Second Maximization: For the second maximization in (59), we define the event $\mathcal{B} \triangleq \{N_{\tilde{\mathbf{x}}_1 \mathbf{y}}(\tilde{P}'_{X_1 X_2 Y}) > 0\}$, yielding

$$\mathbb{P}[\mathcal{B}] \leq M_2 e^{-n I_{\tilde{P}'}(X_2; X_1, Y)}, \quad (62)$$

which follows from the union bound and (37). Whenever $R_2 < I_{\tilde{P}'}(X_2; X_1, Y) + \delta$, we have

$$\begin{aligned} & \mathbb{P}[\mathcal{E}_{P, \tilde{P}}(\tilde{P}'_{X_1 X_2 Y})] \\ & \leq \mathbb{P}[\mathcal{E}_{P, \tilde{P}}(\tilde{P}'_{X_1 X_2 Y}) \mid \mathcal{B}^c] + \mathbb{P}[\mathcal{B}] \mathbb{P}[\mathcal{E}_{P, \tilde{P}}(\tilde{P}'_{X_1 X_2 Y}) \mid \mathcal{B}] \end{aligned} \quad (63)$$

$$\leq \mathbb{1} \left\{ q^n(\tilde{P}_{X_1 X_2 Y}) \geq \underline{G}_{\delta, n}(P_{X_1 X_2 Y}) \right\} + M_2 e^{-n I_{\tilde{P}'}(X_2; X_1, Y)} \mathbb{P}[\mathcal{E}_{P, \tilde{P}}(\tilde{P}'_{X_1 X_2 Y}) \mid \mathcal{B}], \quad (64)$$

$$\begin{aligned} & \leq \mathbb{1} \left\{ q^n(\tilde{P}_{X_1 X_2 Y}) \geq \underline{G}_{\delta, n}(P_{X_1 X_2 Y}) \right\} \\ & \quad + M_2 e^{-n I_{\tilde{P}'}(X_2; X_1, Y)} \mathbb{1} \left\{ q^n(\tilde{P}_{X_1 X_2 Y}) + p_0(n) e^{n 2\delta} q^n(\tilde{P}'_{X_1 X_2 Y}) \geq \underline{G}_{\delta, n}(P_{X_1 X_2 Y}) \right\}, \end{aligned} \quad (65)$$

where (64) follows using (62) and (58) along with the fact that \mathcal{B}^c implies $N_{\tilde{\mathbf{x}}_1 \mathbf{y}}(\tilde{P}'_{X_1 X_2 Y}) = 0$, and (65) follows by conditioning on $\mathcal{A}_\delta(\tilde{P}_{X_1 Y})$ and using (39).

Step 6: Deduce the Exponent for Fixed (x_1, x_2, y)

Observe that $\underline{F}(P_{X_1 X_2 Y}, R_2)$ in (14) equals the exponent of $\underline{G}_{\delta, n}$ in (51) in the limit as $\delta \rightarrow 0$ and $n \rightarrow \infty$. Similarly, the exponents corresponding to the other quantities appearing in the indicator functions in (61) and (65) tend toward the following:

$$\begin{aligned} \overline{F}_1(\tilde{P}_{X_1 X_2 Y}, \tilde{P}'_{X_1 X_2 Y}, R_2) & \triangleq \max \left\{ \mathbb{E}_{\tilde{P}}[\log q(X_1, X_2, Y)], \right. \\ & \quad \left. \mathbb{E}_{\tilde{P}'}[\log q(X_1, X_2, Y)] + R_2 - I_{\tilde{P}'}(X_2; X_1, Y) \right\} \end{aligned} \quad (66)$$

$$\overline{F}_2(\tilde{P}_{X_1 X_2 Y}, \tilde{P}'_{X_1 X_2 Y}) \triangleq \max \left\{ \mathbb{E}_{\tilde{P}}[\log q(X_1, X_2, Y)], \mathbb{E}_{\tilde{P}'}[\log q(X_1, X_2, Y)] \right\}. \quad (67)$$

We claim that combining these expressions with (57), (59), (61) and (65) and taking $\delta \rightarrow 0$ (e.g., analogously to [4, p. 737], we may set $\delta = n^{-1/2}$), gives the following:

$$\bar{p}_{e,1}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}) \leq \max \left\{ \max_{(\tilde{P}_{X_1 X_2 Y}, \tilde{P}'_{X_1 X_2 Y}) \in \mathcal{T}_1^{(1)}(P_{X_1 X_2 Y}, R_2)} M_1 e^{-n I_{\tilde{P}}(X_1; X_2, Y)}, \right. \\ \left. \max_{(\tilde{P}_{X_1 X_2 Y}, \tilde{P}'_{X_1 X_2 Y}) \in \mathcal{T}_1^{(2)}(P_{X_1 X_2 Y}, R_2)} M_1 e^{-n I_{\tilde{P}}(X_1; X_2, Y)} M_2 e^{-n I_{\tilde{P}'}(X_2; X_1, Y)} \right\}, \quad (68)$$

where¹

$$\mathcal{T}_1^{(1)}(P_{X_1 X_2 Y}, R_2) \triangleq \left\{ (\tilde{P}_{X_1 X_2 Y}, \tilde{P}'_{X_1 X_2 Y}) : \tilde{P}_{X_1 X_2 Y} \in \mathcal{S}_1(Q_1, P_{X_2 Y}), \right. \\ \left. \tilde{P}'_{X_1 X_2 Y} \in \mathcal{S}'_1(Q_2, \tilde{P}_{X_1 Y}), I_{\tilde{P}'}(X_2; X_1, Y) \leq R_2, \bar{F}_1(\tilde{P}_{X_1 X_2 Y}, \tilde{P}'_{X_1 X_2 Y}, R_2) \geq \underline{F}(P_{X_1 X_2 Y}, R_2) \right\} \quad (69)$$

$$\mathcal{T}_1^{(2)}(P_{X_1 X_2 Y}, R_2) \triangleq \left\{ (\tilde{P}_{X_1 X_2 Y}, \tilde{P}'_{X_1 X_2 Y}) : \tilde{P}_{X_1 X_2 Y} \in \mathcal{S}_1(Q_1, P_{X_2 Y}), \right. \\ \left. \tilde{P}'_{X_1 X_2 Y} \in \mathcal{S}'_1(Q_2, \tilde{P}_{X_1 Y}), I_{\tilde{P}'}(X_2; X_1, Y) \geq R_2, \bar{F}_2(\tilde{P}_{X_1 X_2 Y}, \tilde{P}'_{X_1 X_2 Y}) \geq \underline{F}(P_{X_1 X_2 Y}, R_2) \right\}, \quad (70)$$

and

$$\mathcal{S}_1(Q_1, P_{X_2 Y}) \triangleq \left\{ \tilde{P}_{X_1 X_2 Y} \in \mathcal{P}(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y}) : \tilde{P}_{X_1} = Q_1, \tilde{P}_{X_2 Y} = P_{X_2 Y} \right\} \quad (71)$$

$$\mathcal{S}'_1(Q_2, \tilde{P}_{X_1 Y}) \triangleq \left\{ \tilde{P}'_{X_1 X_2 Y} \in \mathcal{P}(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y}) : \tilde{P}'_{X_1 Y} = \tilde{P}_{X_1 Y}, \tilde{P}'_{X_2} = Q_2 \right\}. \quad (72)$$

To see that this is true, we note the following:

- For the first term in the maximization in (68), the objective function follows from (56), and the additional constraint $\bar{F}_1(\tilde{P}_{X_1 X_2 Y}, \tilde{P}'_{X_1 X_2 Y}, R_2) \geq \underline{F}(P_{X_1 X_2 Y}, R_2)$ in (69) follows since the left-hand side in (61) has exponent \bar{F}_1 and the right-hand side has exponent \underline{F} by the definition of $\underline{G}_{\delta,n}$ in (51).
- For the second term in the maximization in (68), the objective function follows from (56) and the second term in (65), and the latter (along with $\underline{G}_{\delta,n}$ in (51)) also leads to the final constraint in (70).
- The first term in (65) is upper bounded by the right-hand side of (61), and we already analyzed the latter in order to obtain the first term in (68). Hence, this term can safely be ignored.

Step 7: Deduce the Achievable Rate Region

By a standard property of types [21, Ch. 2], $\mathbb{P}[(\mathbf{X}_1, \mathbf{X}_2, \mathbf{Y}) \in T^n(P_{X_1 X_2 Y})]$ decays to zero exponentially fast when $P_{X_1 X_2 Y}$ is bounded away from $Q_1 \times Q_2 \times W$. Therefore, we can safely substitute $P_{X_1 X_2 Y} = Q_1 \times Q_2 \times W$ to obtain the following rate conditions for the first decoding step:

$$R_1 \leq \min_{(\tilde{P}_{X_1 X_2 Y}, \tilde{P}'_{X_1 X_2 Y}) \in \mathcal{T}_1^{(1)}(Q_1 \times Q_2 \times W, R_2)} I_{\tilde{P}}(X_1; X_2, Y) \quad (73)$$

$$R_1 + R_2 \leq \min_{(\tilde{P}_{X_1 X_2 Y}, \tilde{P}'_{X_1 X_2 Y}) \in \mathcal{T}_1^{(2)}(Q_1 \times Q_2 \times W, R_2)} I_{\tilde{P}}(X_1; X_2, Y) + I_{\tilde{P}'}(X_2; X_1, Y). \quad (74)$$

¹Strictly speaking, these sets depend on (Q_1, Q_2) , but this dependence need not be explicit, since we have $P_{X_1} = Q_1$ and $P_{X_2} = Q_2$.

Finally, we claim that (73)–(74) can be united to obtain (18). To see this, we consider two cases:

- If $R_2 \geq I_{\tilde{P}'}(X_2; X_1, Y)$, then the $[\cdot]^+$ term in (18) equals zero, yielding the objective in (73). Similarly, in this case, the term \bar{F} in (13) simplifies to \bar{F}_1 in (66).
- If $R_2 \leq I_{\tilde{P}'}(X_2; X_1, Y)$, then the $[\cdot]^+$ term in (18) equals $I_{\tilde{P}'}(X_2; X_1, Y) - R_2$, yielding the objective in (73). In this case, the term \bar{F} in (13) simplifies to \bar{F}_2 in (67).

IV. PROOF OF THEOREM 2

The achievability and ensemble tightness proofs follow similar steps. Thus, and since we have already given an achievability proof in Section III, we focus our attention here on the ensemble tightness proof.

Step 1: Initial Bound

We consider the two error events introduced at the beginning of Section III, and observe that $\bar{p}_e \geq \frac{1}{2} \max\{\bar{p}_{e,1}, \bar{p}_{e,2}\}$. The analysis of $\bar{p}_{e,2}$ is precisely that given in [4, Thm. 1], so we focus on $\bar{p}_{e,1}$.

We assume without loss of generality that $m_1 = m_2 = 1$, and we write $\mathbf{X}_\nu = \mathbf{X}_\nu^{(1)}$ ($\nu = 1, 2$), let $\mathbf{X}_2^{(j)}$ denote $\mathbf{X}_2^{(1,j)}$, let $\bar{\mathbf{X}}_2^{(j)}$ denote $\mathbf{X}_2^{(i,j)}$ for some fixed $i \neq 1$, and let $(\bar{\mathbf{X}}_1, \bar{\mathbf{X}}_2)$ denote $(\mathbf{X}_1^{(i)}, \mathbf{X}_2^{(i,j)})$ for some fixed (i, j) with $i \neq 1$. Thus,

$$(\mathbf{X}_1, \mathbf{X}_2, \mathbf{Y}, \bar{\mathbf{X}}_1, \bar{\mathbf{X}}_2) \sim P_{\mathbf{X}_1}(\mathbf{x}_1)P_{\mathbf{X}_2|\mathbf{X}_1}(\mathbf{x}_2|\mathbf{x}_1)W^n(\mathbf{y}|\mathbf{x}_1, \mathbf{x}_2)P_{\mathbf{X}_1}(\bar{\mathbf{x}}_1)P_{\mathbf{X}_2|\mathbf{X}_1}(\bar{\mathbf{x}}_2|\bar{\mathbf{x}}_1). \quad (75)$$

Moreover, analogously to (34), we define

$$\Xi_{\mathbf{x}_2\mathbf{y}}(\mathbf{x}_1) \triangleq q^n(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}) + \sum_{j \neq 1} q^n(\mathbf{x}_1, \mathbf{X}_2^{(1,j)}, \mathbf{y}) \quad (76)$$

$$\tilde{\Xi}_{\mathbf{y}}(\mathbf{x}_1^{(i)}) \triangleq \sum_j q^n(\mathbf{x}_1^{(i)}, \mathbf{X}_2^{(i,j)}, \mathbf{y}). \quad (77)$$

Not that here we use separate definitions corresponding to \mathbf{x}_1 and $\mathbf{x}_1^{(i)}$ ($i \neq 1$) since in the cognitive MAC, each user-1 sequence is associated with a different set of user-2 sequences.

Fix a joint type $P_{X_1 X_2 Y}$ and a triplet $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}) \in T^n(P_{X_1 X_2 Y})$, and let $\bar{p}_{e,1}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y})$ be the type-1 error probability conditioned on $(\mathbf{X}_1^{(1)}, \mathbf{X}_2^{(1,1)}, \mathbf{Y}) = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{y})$; here we assume without loss of generality that $m_1 = m_2 = 1$. We have

$$\bar{p}_{e,1}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}) = \mathbb{P} \left[\bigcup_{i=2}^{M_1} \left\{ \tilde{\Xi}_{\mathbf{y}}(\mathbf{X}_1^{(i)}) \geq \Xi_{\mathbf{x}_2\mathbf{y}}(\mathbf{x}_1) \right\} \right] \quad (78)$$

$$\geq \frac{1}{2} \min \left\{ 1, (M_1 - 1) \mathbb{P}[\tilde{\Xi}_{\mathbf{y}}(\bar{\mathbf{X}}_1) \geq \Xi_{\mathbf{x}_2\mathbf{y}}(\mathbf{x}_1)] \right\}, \quad (79)$$

where (79) follows since the truncated union bound is tight to within a factor of $\frac{1}{2}$ for independent events [25, Lemma A.2]. Note that this argument fails for the standard MAC; there, the independence requirement does not hold, so it is unclear whether (35) is tight upon taking the minimum with 1.

We now bound the inner probability in (79), which we denote by $\Phi_1(P_{X_1 X_2 Y})$. By similarly defining

$$\Phi_2(P_{X_1 X_2 Y}, \tilde{P}_{X_1 Y}) \triangleq \mathbb{P}[\tilde{\Xi}_{\mathbf{y}}(\bar{\mathbf{X}}_1) \geq \Xi_{\mathbf{x}_2\mathbf{y}}(\mathbf{x}_1) | (\bar{\mathbf{X}}_1, \mathbf{y}) \in T^n(\tilde{P}_{X_1 Y})], \quad (80)$$

we obtain

$$\Phi_1(P_{X_1 X_2 Y}) \geq \max_{\tilde{P}_{X_1 Y}} \mathbb{P}[(\bar{\mathbf{X}}_1, \mathbf{y}) \in T^n(\tilde{P}_{X_1 Y})] \Phi_2(P_{X_1 X_2 Y}, \tilde{P}_{X_1 Y}) \quad (81)$$

$$\stackrel{\doteq}{=} \max_{\tilde{P}_{X_1 Y} : \tilde{P}_{X_1} = Q_{X_1}, \tilde{P}_Y = P_Y} e^{-n I_{\tilde{P}}(X_1; Y)} \Phi_2(P_{X_1 X_2 Y}, \tilde{P}_{X_1 Y}), \quad (82)$$

where (82) is a standard property of types [21, Ch. 2]. We proceed by bounding Φ_2 ; to do so, we let $\bar{\mathbf{x}}_1$ be an arbitrary sequence such that $(\bar{\mathbf{x}}_1, \mathbf{y}) \in T^n(\tilde{P}_{X_1 Y})$. By symmetry, any such sequence may be considered.

Step 2: Type Class Enumerators

We write each metric $\Xi_{\mathbf{x}_2 \mathbf{y}}$ in terms of type class enumerators. Specifically, again writing $q^n(P_{X_1 X_2 Y})$ to denote the n -fold product metric for a given joint type, we note the following analogs of (46):

$$\Xi_{\mathbf{x}_2 \mathbf{y}}(\mathbf{x}_1) = q^n(P_{X_1 X_2 Y}) + \sum_{P'_{X_1 X_2 Y}} \Xi_{\mathbf{y}}(\mathbf{x}_1, P'_{X_1 X_2 Y}) \quad (83)$$

$$\tilde{\Xi}_{\mathbf{y}}(\bar{\mathbf{x}}_1) = \sum_{\tilde{P}'_{X_1 X_2 Y}} \tilde{\Xi}_{\mathbf{y}}(\bar{\mathbf{x}}_1, \tilde{P}'_{X_1 X_2 Y}), \quad (84)$$

where

$$\Xi_{\mathbf{y}}(\mathbf{x}_1, P'_{X_1 X_2 Y}) \triangleq N_{\mathbf{x}_1 \mathbf{y}}(P'_{X_1 X_2 Y}) q^n(P'_{X_1 X_2 Y}) \quad (85)$$

$$\tilde{\Xi}_{\mathbf{y}}(\bar{\mathbf{x}}_1, \tilde{P}'_{X_1 X_2 Y}) \triangleq \tilde{N}_{\bar{\mathbf{x}}_1 \mathbf{y}}(\tilde{P}'_{X_1 X_2 Y}) q^n(\tilde{P}'_{X_1 X_2 Y}), \quad (86)$$

and

$$N_{\mathbf{x}_1 \mathbf{y}}(P'_{X_1 X_2 Y}) \triangleq \sum_{j \neq 1} \mathbb{1}\{(\mathbf{x}_1, \mathbf{X}_2^{(j)}, \mathbf{y}) \in T^n(P'_{X_1 X_2 Y})\} \quad (87)$$

$$\tilde{N}_{\bar{\mathbf{x}}_1 \mathbf{y}}(\tilde{P}'_{X_1 X_2 Y}) \triangleq \sum_j \mathbb{1}\{(\bar{\mathbf{x}}_1, \bar{\mathbf{X}}_2^{(j)}, \mathbf{y}) \in T^n(\tilde{P}'_{X_1 X_2 Y})\}. \quad (88)$$

Note the minor differences in these definitions compared to those for the standard MAC, resulting from the differing codebook structure associated with superposition coding. Using these definitions, we can bound (80) as follows:

$$\Phi_2(P_{X_1 X_2 Y}, \tilde{P}_{X_1 Y}) = \mathbb{P}\left[\sum_{\tilde{P}'_{X_1 X_2 Y}} \tilde{\Xi}_{\mathbf{y}}(\bar{\mathbf{x}}_1, \tilde{P}'_{X_1 X_2 Y}) \geq q^n(P_{X_1 X_2 Y}) + \sum_{P'_{X_1 X_2 Y}} \Xi_{\mathbf{y}}(\mathbf{x}_1, P'_{X_1 X_2 Y})\right] \quad (89)$$

$$\geq \mathbb{P}\left[\max_{\tilde{P}'_{X_1 X_2 Y}} \tilde{\Xi}_{\mathbf{y}}(\bar{\mathbf{x}}_1, \tilde{P}'_{X_1 X_2 Y}) \geq q^n(P_{X_1 X_2 Y}) + p_0(n) \max_{P'_{X_1 X_2 Y}} \Xi_{\mathbf{y}}(\mathbf{x}_1, P'_{X_1 X_2 Y})\right] \quad (90)$$

$$\geq \max_{\tilde{P}'_{X_1 X_2 Y}} \mathbb{P}\left[\tilde{\Xi}_{\mathbf{y}}(\bar{\mathbf{x}}_1, \tilde{P}'_{X_1 X_2 Y}) \geq q^n(P_{X_1 X_2 Y}) + p_0(n) \max_{P'_{X_1 X_2 Y}} \Xi_{\mathbf{y}}(\mathbf{x}_1, P'_{X_1 X_2 Y})\right] \quad (91)$$

$$\triangleq \max_{\tilde{P}'_{X_1 X_2 Y}} \Phi_3(P_{X_1 X_2 Y}, \tilde{P}_{X_1 Y}, \tilde{P}'_{X_1 X_2 Y}), \quad (92)$$

where $p_0(n)$ is a polynomial corresponding to the number of joint types.

Step 3: An Auxiliary Lemma

We define the sets

$$\mathcal{S}_{1c,n}(Q_{X_1,n}, P_Y) \triangleq \left\{ \tilde{P}_{X_1Y} \in \mathcal{P}_n(\mathcal{X}_1 \times \mathcal{Y}) : \tilde{P}_{X_1} = Q_{X_1,n}, \tilde{P}_Y = P_Y \right\} \quad (93)$$

$$\mathcal{S}'_{1c,n}(Q_{X_1X_2,n}, \tilde{P}_{X_1Y}) \triangleq \left\{ \tilde{P}'_{X_1X_2Y} \in \mathcal{P}_n(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y}) : \tilde{P}'_{X_1Y} = \tilde{P}_{X_1Y}, \tilde{P}'_{X_1X_2} = Q_{X_1X_2,n} \right\}. \quad (94)$$

The following lemma provides analogous properties to Lemma 1 for joint types within $\mathcal{S}'_{1c,n}$, with suitable modifications to handle the fact that we are proving ensemble tightness rather than achievability. It is based on the fact that $N_{\bar{\mathbf{x}}_1\mathbf{y}}(\tilde{P}'_{X_1X_2Y})$ has a binomial distribution with success probability $\mathbb{P}[(\bar{\mathbf{x}}_1, \bar{\mathbf{X}}_2, \mathbf{y}) \in T^n(\tilde{P}'_{X_1X_2Y}) | \bar{\mathbf{X}}_1 = \bar{\mathbf{x}}_1] \doteq e^{-nI_{\tilde{P}'}(X_2;Y|X_1)}$ by (23).

Lemma 2. Fix a joint type \tilde{P}_{X_1Y} and a pair $(\bar{\mathbf{x}}_1, \mathbf{y}) \in T^n(\tilde{P}_{X_1Y})$. For any joint type $\tilde{P}'_{X_1X_2Y} \in \mathcal{S}'_{1,n}(Q_{X_1X_2,n}, \tilde{P}_{X_2Y})$ and constant $\delta > 0$, the type enumerator $N_{\bar{\mathbf{x}}_1\mathbf{y}}(\tilde{P}'_{X_1X_2Y})$ satisfies the following:

- 1) If $R_2 \geq I_{\tilde{P}'}(X_2;Y|X_1) - \delta$, then $N_{\bar{\mathbf{x}}_1\mathbf{y}}(\tilde{P}'_{X_1X_2Y}) \leq M_2 e^{-n(I_{\tilde{P}'}(X_2;Y|X_1) - 2\delta)}$ with probability approaching one super-exponentially fast
- 2) If $R_2 \geq I_{\tilde{P}'}(X_2;Y|X_1) + \delta$, then $N_{\bar{\mathbf{x}}_1\mathbf{y}}(\tilde{P}'_{X_1X_2Y}) \geq M_2 e^{-n(I_{\tilde{P}'}(X_2;Y|X_1) + \delta)}$ with probability approaching one super-exponentially fast
- 3) If $R_2 \leq I_{\tilde{P}'}(X_2;Y|X_1) - \delta$, then
 - a) $N_{\bar{\mathbf{x}}_1\mathbf{y}}(\tilde{P}'_{X_1X_2Y}) \leq e^{n\delta}$ with probability approaching one super-exponentially fast;
 - b) $\mathbb{P}[N_{\bar{\mathbf{x}}_1\mathbf{y}}(\tilde{P}'_{X_1X_2Y}) > 0] \doteq M_2 e^{-nI_{\tilde{P}'}(X_2;Y|X_1)}.$

Moreover, the analogous properties hold for the type enumerator $N_{\mathbf{x}_1\mathbf{y}}(P'_{X_1X_2Y})$ and any joint types P_{X_1Y} (with $P_{X_1} = Q_{X_1,n}$) and $\tilde{P}'_{X_1X_2Y} \in \mathcal{S}'_{1,n}(Q_{X_1X_2,n}, P_{X_1Y})$.

Proof: Parts 1, 2 and 3a are proved in the same way as Lemma 1; we omit the details to avoid repetition with [12], [22]. Part 3b follows by writing the probability that $N_{\bar{\mathbf{x}}_1\mathbf{y}} > 0$ as a union of the $M_1 - 1$ events in (87) holding, and using the fact that the truncated union bound is tight to within a factor of $\frac{1}{2}$ [25, Lemma A.2]. The truncation need not explicitly be included, since the assumption of part 3 implies that $M_2 e^{-nI_{\tilde{P}'}(X_2;Y|X_1)} \rightarrow 0$. ■

Given a joint type P_{X_2Y} (respectively, \tilde{P}_{X_1Y}), let $\mathcal{A}_\delta(\tilde{P}_{X_1Y})$ (respectively, $\tilde{\mathcal{A}}_\delta(\tilde{P}_{X_1Y})$) denote the union of the high-probability events in Lemma 2 (in parts 1, 2 and 3a) taken over all $P'_{X_1X_2Y} \in \mathcal{S}_{1,n}(Q_{X_1X_2}, P_{X_2Y})$ (respectively, $\tilde{P}'_{X_1X_2Y} \in \mathcal{S}'_{1,n}(Q_{X_1X_2}, \tilde{P}_{X_1Y})$). By the union bound, the probability of these events tends to one super-exponentially fast, and hence we can safely condition any event accordingly without changing the exponential behavior of the corresponding probability (see (41)–(45)).

Step 4: Bound $\Xi_{\mathbf{y}}(\mathbf{x}_1, P'_{X_1X_2Y})$ by a Deterministic Quantity

We first deal with $\Xi_{\mathbf{y}}(\mathbf{x}_1, P'_{X_1X_2Y})$ in (91). Defining the event

$$\mathcal{B}_\delta \triangleq \left\{ N_{\mathbf{x}_1\mathbf{y}}(P'_{X_1X_2Y}) = 0 \text{ for all } P'_{X_1X_2Y} \text{ such that } R_2 \leq I_{\tilde{P}'}(X_2;Y|X_1) - \delta \right\}, \quad (95)$$

we immediately obtain from Property 3b in Lemma 2 that $\mathbb{P}[\mathcal{B}_\delta^c] \leq e^{-n\delta} \rightarrow 0$, and hence

$$\Phi_3(P_{X_1 X_2 Y}, \tilde{P}_{X_1 Y}, \tilde{P}'_{X_1 X_2 Y}) \geq \mathbb{P} \left[\tilde{\Xi}_{\mathbf{y}}(\bar{\mathbf{x}}_1, \tilde{P}'_{X_1 X_2 Y}) \geq q^n(P_{X_1 X_2 Y}) + p_0(n) \max_{P'_{X_1 X_2 Y}} \Xi_{\mathbf{y}}(\mathbf{x}_1, P'_{X_1 X_2 Y}) \cap \mathcal{B}_\delta \right] \quad (96)$$

$$\doteq \mathbb{P} \left[\tilde{\Xi}_{\mathbf{y}}(\bar{\mathbf{x}}_1, \tilde{P}'_{X_1 X_2 Y}) \geq q^n(P_{X_1 X_2 Y}) + p_0(n) \max_{P'_{X_1 X_2 Y}} \Xi_{\mathbf{y}}(\mathbf{x}_1, P'_{X_1 X_2 Y}) \mid \mathcal{B}_\delta \right]. \quad (97)$$

Next, conditioned on both \mathcal{B}_δ and the events in Lemma 2, we have

$$q^n(P_{X_1 X_2 Y}) + p_0(n) \max_{P'_{X_1 X_2 Y}} \Xi_{\mathbf{y}}(\mathbf{x}_1, P'_{X_1 X_2 Y}) \quad (98)$$

$$= q^n(P_{X_1 X_2 Y}) + p_0(n) \max_{\substack{P'_{X_1 X_2 Y} \in \mathcal{S}'_{1c,n}(Q_{X_1 X_2}, n, \tilde{P}_{X_1 Y}) : \\ R_2 \geq I_{P'}(X_2; Y|X_1) - \delta}} \Xi_{\mathbf{y}}(\mathbf{x}_1, P'_{X_1 X_2 Y}) \quad (99)$$

$$\leq q^n(P_{X_1 X_2 Y}) + p_0(n) \max_{\substack{P'_{X_1 X_2 Y} \in \mathcal{S}'_{1c,n}(Q_{X_1 X_2}, n, \tilde{P}_{X_1 Y}) : \\ R_2 \geq I_{P'}(X_2; Y|X_1) - \delta}} M_2 e^{-n(I_{\tilde{P}'}(X_2; Y|X_1) - 2\delta)} q^n(P'_{X_1 X_2 Y}) \quad (100)$$

$$\triangleq \bar{G}_{\delta,n}(P_{X_1 X_2 Y}), \quad (101)$$

where in (100) we used part 1 of Lemma 2. It follows that

$$\Phi_3(P_{X_1 X_2 Y}, \tilde{P}_{X_1 Y}, \tilde{P}'_{X_1 X_2 Y}) \geq \mathbb{P}[\tilde{\Xi}_{\mathbf{y}}(\bar{\mathbf{x}}_1, \tilde{P}'_{X_1 X_2 Y}) \geq \bar{G}_{\delta,n}(P_{X_1 X_2 Y})], \quad (102)$$

where the conditioning on \mathcal{B}_δ has been removed since it is independent of the statistics of $\tilde{\Xi}_{\mathbf{y}}(\bar{\mathbf{x}}_1, \tilde{P}'_{X_1 X_2 Y})$.

Step 5: Bound $\Xi_{x_2 \mathbf{y}}(\bar{\mathbf{x}}_1)$ by a Deterministic Quantity

We now deal with $\tilde{\Xi}_{\mathbf{y}}(\bar{\mathbf{x}}_1, \tilde{P}'_{X_1 X_2 Y})$. Substituting (102) into (92) and constraining the maximization in two different ways, we obtain

$$\Phi_2(P_{X_1 X_2 Y}, \tilde{P}_{X_1 Y}) \geq \max \left\{ \max_{\substack{\tilde{P}'_{X_1 X_2 Y} \in \mathcal{S}'_{1c,n}(Q_{X_1 X_2}, n, \tilde{P}_{X_1 Y}) : \\ R_2 \geq I_{\tilde{P}'}(X_2; Y|X_1) + \delta}} \mathbb{P} \left[\tilde{\Xi}_{\mathbf{y}}(\bar{\mathbf{x}}_1, \tilde{P}'_{X_1 X_2 Y}) \geq \bar{G}_{\delta,n}(P_{X_1 X_2 Y}) \right], \right. \\ \left. \max_{\substack{\tilde{P}'_{X_1 X_2 Y} \in \mathcal{S}'_{1c,n}(Q_{X_1 X_2}, n, \tilde{P}_{X_1 Y}) : \\ R_2 \leq I_{\tilde{P}'}(X_2; Y|X_1) - \delta}} \mathbb{P} \left[\tilde{\Xi}_{\mathbf{y}}(\bar{\mathbf{x}}_1, \tilde{P}'_{X_1 X_2 Y}) \geq \bar{G}_{\delta,n}(P_{X_1 X_2 Y}) \right] \right\}. \quad (103)$$

For $R_2 \geq I_{\tilde{P}'}(X_2; Y|X_1) + \delta$, we have from part 2 of Lemma 2 that, conditioned on $\tilde{\mathcal{A}}_\delta(\tilde{P}_{X_1 Y})$,

$$\tilde{\Xi}_{\mathbf{y}}(\bar{\mathbf{x}}_1, \tilde{P}'_{X_1 X_2 Y}) \geq M_2 e^{-n(I_{\tilde{P}'}(X_2; Y|X_1) + \delta)} q^n(\tilde{P}'_{X_1 X_2 Y}). \quad (104)$$

On the other hand, for $R_2 \leq I_{\tilde{P}'}(X_2; Y|X_1) - \delta$, we have

$$\mathbb{P}\left[\tilde{\Xi}_{\mathbf{y}}(\bar{\mathbf{x}}_1, \tilde{P}'_{X_1 X_2 Y}) \geq \bar{G}_{\delta, n}(P_{X_1 X_2 Y})\right] \quad (105)$$

$$= \mathbb{P}\left[\tilde{\Xi}_{\mathbf{y}}(\bar{\mathbf{x}}_1, \tilde{P}'_{X_1 X_2 Y}) \geq \bar{G}_{\delta, n}(P_{X_1 X_2 Y}) \cap N_{\bar{\mathbf{x}}_1 \mathbf{y}}(\tilde{P}'_{X_1 X_2 Y}) > 0\right] \quad (106)$$

$$= \mathbb{P}\left[\tilde{\Xi}_{\mathbf{y}}(\bar{\mathbf{x}}_1, \tilde{P}'_{X_1 X_2 Y}) \geq \bar{G}_{\delta, n}(P_{X_1 X_2 Y}) \mid N_{\bar{\mathbf{x}}_1 \mathbf{y}}(\tilde{P}'_{X_1 X_2 Y}) > 0\right] \mathbb{P}\left[N_{\bar{\mathbf{x}}_1 \mathbf{y}}(\tilde{P}'_{X_1 X_2 Y}) > 0\right] \quad (107)$$

$$\doteq \mathbb{P}\left[\tilde{\Xi}_{\mathbf{y}}(\bar{\mathbf{x}}_1, \tilde{P}'_{X_1 X_2 Y}) \geq \bar{G}_{\delta, n}(P_{X_1 X_2 Y}) \mid N_{\bar{\mathbf{x}}_1 \mathbf{y}}(\tilde{P}'_{X_1 X_2 Y}) > 0\right] M_2 e^{-n I_{\tilde{P}'}(X_2; Y|X_1)} \quad (108)$$

$$\doteq \mathbb{1}\left\{q^n(\tilde{P}'_{X_1 X_2 Y}) \geq \bar{G}_{\delta, n}(P_{X_1 X_2 Y})\right\} M_2 e^{-n I_{\tilde{P}'}(X_2; Y|X_1)}, \quad (109)$$

where (106) follows since the event under consideration is zero unless $N_{\bar{\mathbf{x}}_1 \mathbf{y}}(\tilde{P}'_{X_1 X_2 Y}) > 0$, (108) follows from part 3b of Lemma 2, and (109) follows since when $N_{\bar{\mathbf{x}}_1 \mathbf{y}}(\tilde{P}'_{X_1 X_2 Y})$ is positive it must be at least one.

Step 6: Deduce the Exponent for Fixed $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y})$

We have now handled both cases in (103). Combining them, and substituting the result into (82), we obtain

$$\begin{aligned} \Phi_1(P_{X_1 X_2 Y}) &\stackrel{\cdot}{\geq} \max_{\tilde{P}_{X_1 Y} \in \mathcal{S}_{1c, n}(Q_{X_1, n}, P_Y)} e^{-n I_{\tilde{P}}(X_1; Y)} \\ &\max \left\{ \max_{\substack{\tilde{P}'_{X_1 X_2 Y} \in \mathcal{S}'_{1c, n}(Q_{X_1 X_2, n}, \tilde{P}_{X_1 Y}) : \\ R_2 \geq I_{\tilde{P}'}(X_2; Y|X_1) + \delta}} \mathbb{1}\left\{M_2 e^{-n(I_{\tilde{P}'}(X_2; Y|X_1) + \delta)} q^n(\tilde{P}'_{X_1 X_2 Y}) \geq \bar{G}_{\delta, n}(P_{X_1 X_2 Y})\right\}, \right. \\ &\quad \left. \max_{\substack{\tilde{P}'_{X_1 X_2 Y} \in \mathcal{S}'_{1c, n}(Q_{X_1 X_2, n}, \tilde{P}_{X_1 Y}) : \\ R_2 \leq I_{\tilde{P}'}(X_2; Y|X_1) - \delta}} M_2 e^{-n I_{\tilde{P}'}(X_2; Y|X_1)} \mathbb{1}\left\{q^n(\tilde{P}'_{X_1 X_2 Y}) \geq \bar{G}_{\delta, n}(P_{X_1 X_2 Y})\right\} \right\}. \end{aligned} \quad (110)$$

Observe that $\bar{F}_c(P_{X_1 X_2 Y})$ in (14) equals the exponent of $\bar{G}_{\delta, n}$ in (101) in the limit as $\delta \rightarrow 0$ and $n \rightarrow \infty$. Similarly, the exponent corresponding to the quantity in the first indicator function in (110) tends to

$$\bar{F}_{1c}(\tilde{P}'_{X_1 X_2 Y}, R_2) \triangleq \mathbb{E}_{\tilde{P}'}[\log q(X_1, X_2, Y)] + R_2 - I_{\tilde{P}'}(X_2; Y|X_1). \quad (111)$$

Recalling that Φ_1 is the inner probability in (79), we obtain the following by taking $\delta \rightarrow 0$ (e.g., $\delta = n^{1/2}$) and using the continuity of the underlying terms in the optimizations:

$$\begin{aligned} \bar{p}_{e,1}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}) &\stackrel{\cdot}{\geq} \max \left\{ \max_{(\tilde{P}_{X_1 Y}, \tilde{P}'_{X_1 X_2 Y}) \in \mathcal{T}_{1c}^{(1)}(P_{X_1 X_2 Y}, R_2)} M_1 e^{-n I_{\tilde{P}}(X_1; Y)}, \right. \\ &\quad \left. \max_{(\tilde{P}_{X_1 Y}, \tilde{P}'_{X_1 X_2 Y}) \in \mathcal{T}_{1c}^{(2)}(P_{X_1 X_2 Y}, R_2)} M_1 e^{-n I_{\tilde{P}}(X_1; Y)} M_2 e^{-n I_{\tilde{P}'}(X_2; Y|X_1)} \right\}, \end{aligned} \quad (112)$$

where

$$\begin{aligned} \mathcal{T}_{1c}^{(1)}(P_{X_1 X_2 Y}, R_2) &\triangleq \left\{(\tilde{P}_{X_1 Y}, \tilde{P}'_{X_1 X_2 Y}) : \tilde{P}_{X_1 Y} \in \mathcal{S}_{1c}(Q_{X_1}, P_Y), \right. \\ &\quad \left. \tilde{P}'_{X_1 X_2 Y} \in \mathcal{S}'_{1c}(Q_{X_1 X_2}, \tilde{P}_{X_1 Y}), I_{\tilde{P}'}(X_2; Y|X_1) \leq R_2, \bar{F}_{1c}(\tilde{P}'_{X_1 X_2 Y}, R_2) \geq \underline{F}_c(P_{X_1 X_2 Y}, R_2)\right\} \end{aligned} \quad (113)$$

$$\begin{aligned} \mathcal{T}_{1c}^{(2)}(P_{X_1 X_2 Y}, R_2) &\triangleq \left\{(\tilde{P}_{X_1 Y}, \tilde{P}'_{X_1 X_2 Y}) : \tilde{P}_{X_1 Y} \in \mathcal{S}_{1c}(Q_{X_1}, P_Y), \right. \\ &\quad \left. \tilde{P}'_{X_1 X_2 Y} \in \mathcal{S}'_{1c}(Q_{X_1 X_2}, \tilde{P}_{X_1 Y}), I_{\tilde{P}'}(X_2; Y|X_1) \geq R_2, \mathbb{E}_{\tilde{P}'}[\log q(X_1, X_2, Y)] \geq \underline{F}(P_{X_1 X_2 Y}, R_2)\right\}, \end{aligned} \quad (114)$$

and

$$\mathcal{S}_{1c}(Q_{X_1}, P_Y) \triangleq \left\{ \tilde{P}_{X_1 X_2 Y} \in \mathcal{P}(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y}) : \tilde{P}_{X_1} = Q_{X_1}, \tilde{P}_Y = P_Y \right\} \quad (115)$$

$$\mathcal{S}'_{1c}(Q_{X_1 X_2}, \tilde{P}_{X_1 Y}) \triangleq \left\{ \tilde{P}'_{X_1 X_2 Y} \in \mathcal{P}(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y}) : \tilde{P}'_{X_1 Y} = \tilde{P}_{X_1 Y}, \tilde{P}'_{X_1 X_2} = Q_{X_1 X_2} \right\}. \quad (116)$$

Specifically, this follows from the same argument as Step 6 in Section III.

Step 7: Deduce the Achievable Rate Region

Similarly to the achievability proof in Section III, the fact that the joint type of $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{Y})$ approaches $Q_{X_1 X_2} \times W$ with probability approaching one means that we can substitute $P_{X_1 X_2 Y} = Q_{X_1 X_2} \times W$ to obtain the following rate conditions:

$$R_1 \leq \min_{(\tilde{P}_{X_1 Y}, \tilde{P}'_{X_1 X_2 Y}) \in \mathcal{T}_{1c}^{(1)}(Q_{X_1 X_2} \times W, R_2)} I_{\tilde{P}}(X_1; Y) \quad (117)$$

$$R_1 + R_2 \leq \min_{(\tilde{P}_{X_1 Y}, \tilde{P}'_{X_1 X_2 Y}) \in \mathcal{T}_{1c}^{(2)}(Q_{X_1 X_2} \times W, R_2)} I_{\tilde{P}}(X_1; Y) + I_{\tilde{P}'}(X_2; Y|X_1). \quad (118)$$

The proof of (30) is now concluded via the same argument as Step 7 in Section III, using the definitions of \overline{F}_c , \overline{F}_{1c} , \mathcal{S}_{1c} , \mathcal{S}'_{1c} , $\mathcal{T}_{1c}^{(1)}$ and $\mathcal{T}_{1c}^{(2)}$ to unite (117)–(118). Note that the optimization variable $\tilde{P}_{X_1 Y}$ can be absorbed into $\tilde{P}'_{X_1 X_2 Y}$ due to the constraint $\tilde{P}'_{X_1 Y} = \tilde{P}_{X_1 Y}$.

V. CONCLUSION

We have obtained error exponents and achievable rates for both the standard and cognitive MAC using a mismatched multi-letter successive decoding rule. For the cognitive case, we have proved ensemble tightness, thus allowing us to conclusively establish that there are cases in which neither the mismatched successive decoding region nor the mismatched maximum-metric decoding region [3] dominate each other in the random coding setting.

An immediate direction for further work is to establish the ensemble tightness of the achievable rate region for the standard MAC in Theorem 1. A more challenging open question is to determine whether either of the *true* mismatched capacity regions (rather than just achievable random-coding regions) for the two decoding rules contain each other in general.

APPENDIX A

BEHAVIOR OF SUCCESSIVE DECODER WITH $q = W$

Here we show that a rate pair (R_1, R_2) or error exponent $E(R_1, R_2)$ is achievable under maximum-likelihood (ML) decoding if and only if it is achievable under the successive rule in (2)–(3) with $q(x_1, x_2, y) = W(y|x_1, x_2)$. This is shown in the same way for the standard MAC and the cognitive MAC, so we focus on the former.

It suffices to show that, for any fixed codebooks $\mathcal{C}_1 = \{\mathbf{x}_1^{(i)}\}_{i=1}^{M_1}$ and $\mathcal{C}_2 = \{\mathbf{x}_2^{(j)}\}_{j=1}^{M_2}$, the error probability under ML decoding is lower bounded by a constant times the error probability under successive decoding. It also suffices to consider the variations of these decoders where ties are broken as errors, since doing so reduces the error probability by at most a factor of two [24]. Formally, we consider the following:

- 1) The ML decoder maximizing $W^n(\mathbf{y}|\mathbf{x}_1^{(i)}, \mathbf{x}_2^{(j)});$
- 2) The successive decoder in (2)–(3) with $q = W;$
- 3) The genie-aided successive decoder using the true value of \mathbf{m}_1 on the second step rather than $\hat{\mathbf{m}}_1$ [11]:

$$\hat{\mathbf{m}}_1 = \arg \max_i \sum_j W^n(\mathbf{x}_1^{(i)}, \mathbf{x}_2^{(j)}, \mathbf{y}) \quad (119)$$

$$\hat{\mathbf{m}}_2 = \arg \max_j W^n(\mathbf{x}_1^{(\mathbf{m}_1)}, \mathbf{x}_2^{(j)}, \mathbf{y}). \quad (120)$$

We denote the probabilities under these decoders by $\mathbb{P}^{(\text{ML})}[\cdot]$, $\mathbb{P}^{(\text{S})}[\cdot]$ and $\mathbb{P}^{(\text{Genie})}[\cdot]$ respectively. Denoting the random message pair by $(\mathbf{m}_1, \mathbf{m}_2)$, the resulting estimate by $(\hat{\mathbf{m}}_1, \hat{\mathbf{m}}_2)$, and the output sequence by \mathbf{Y} , we have

$$\begin{aligned} & \mathbb{P}^{(\text{ML})}[(\hat{\mathbf{m}}_1, \hat{\mathbf{m}}_2) \neq (\mathbf{m}_1, \mathbf{m}_2)] \\ & \geq \max \left\{ \mathbb{P}^{(\text{ML})}[\hat{\mathbf{m}}_1 \neq \mathbf{m}_1], \mathbb{P}^{(\text{ML})} \left[\bigcup_{j \neq \mathbf{m}_2} \left\{ W^n(\mathbf{x}_1^{(\mathbf{m}_1)}, \mathbf{x}_2^{(j)}, \mathbf{Y}) \geq W^n(\mathbf{x}_1^{(\mathbf{m}_1)}, \mathbf{x}_2^{(\mathbf{m}_2)}, \mathbf{Y}) \right\} \right] \right\} \end{aligned} \quad (121)$$

$$\geq \max \left\{ \mathbb{P}^{(\text{Genie})}[\hat{\mathbf{m}}_1 \neq \mathbf{m}_1], \mathbb{P}^{(\text{Genie})} \left[\bigcup_{j \neq \mathbf{m}_2} \left\{ W^n(\mathbf{x}_1^{(\mathbf{m}_1)}, \mathbf{x}_2^{(j)}, \mathbf{Y}) \geq W^n(\mathbf{x}_1^{(\mathbf{m}_1)}, \mathbf{x}_2^{(\mathbf{m}_2)}, \mathbf{Y}) \right\} \right] \right\} \quad (122)$$

$$\geq \frac{1}{2} \mathbb{P}^{(\text{Genie})}[(\hat{\mathbf{m}}_1, \hat{\mathbf{m}}_2) \neq (\mathbf{m}_1, \mathbf{m}_2)] \quad (123)$$

$$= \frac{1}{2} \mathbb{P}^{(\text{S})}[(\hat{\mathbf{m}}_1, \hat{\mathbf{m}}_2) \neq (\mathbf{m}_1, \mathbf{m}_2)], \quad (124)$$

where (122) follows since the two steps of the genie-aided decoder minimize the two terms in the $\max\{\cdot, \cdot\}$, (123) follows by writing $\max\{\mathbb{P}[A], \mathbb{P}[B]\} \geq \frac{1}{2}(\mathbb{P}[A] + \mathbb{P}[B]) \geq \frac{1}{2}\mathbb{P}[A \cup B]$, and (124) follows since the overall error probability is unchanged by the genie [11].

APPENDIX B

FORMULATIONS OF (18) AND (30) IN TERMS OF CONVEX OPTIMIZATION PROBLEMS

Here we write (18) in terms of convex optimization problems, starting with the alternative formulation in (73)–(74). We first note that (74) holds if and only if

$$R_1 \leq \min_{(\tilde{P}_{X_1 X_2 Y}, \tilde{P}'_{X_1 X_2 Y}) \in \mathcal{T}_1^{(2)}(Q_1 \times Q_2 \times W, R_2)} I_{\tilde{P}}(X_1; X_2, Y) + [I_{\tilde{P}'}(X_2; X_1, Y) - R_2]^+, \quad (125)$$

since the argument to the $[\cdot]^+$ is always non-negative due to the constraint $I_{\tilde{P}'}(X_2; X_1, Y) \geq R_2$. Next, we claim that when combining (73) and (125), the rate region is unchanged if the constraint $I_{\tilde{P}'}(X_2; X_1, Y) \geq R_2$ is omitted from (125). This is seen by noting that whenever $I_{\tilde{P}'}(X_2; X_1, Y) < R_2$, the objective in (125) coincides with that of (73), whereas the latter has a less restrictive constraint since $\bar{F}_1 > \bar{F}_2$ (see (66)–(67)).

We now deal with the non-concavity of the functions \bar{F}_1 and \bar{F}_2 appearing in the sets $\mathcal{T}_1^{(1)}$ and $\mathcal{T}_1^{(2)}$. Using the identity

$$\min_{x \leq \max\{a, b\}} f(x) = \min \left\{ \min_{x \leq a} f(x), \min_{x \leq b} f(x) \right\}, \quad (126)$$

we obtain the following rate conditions from (73) and (125):

$$R_1 \leq \min_{(\tilde{P}_{X_1 X_2 Y}, \tilde{P}'_{X_1 X_2 Y}) \in \mathcal{T}_1^{(1,1)}(Q_1 \times Q_2 \times W, R_2)} I_{\tilde{P}}(X_1; X_2, Y) \quad (127)$$

$$R_1 \leq \min_{(\tilde{P}_{X_1 X_2 Y}, \tilde{P}'_{X_1 X_2 Y}) \in \mathcal{T}_1^{(1,2)}(Q_1 \times Q_2 \times W, R_2)} I_{\tilde{P}}(X_1; X_2, Y) \quad (128)$$

$$R_1 \leq \min_{(\tilde{P}_{X_1 X_2 Y}, \tilde{P}'_{X_1 X_2 Y}) \in \mathcal{T}_1^{(2,1)}(Q_1 \times Q_2 \times W, R_2)} I_{\tilde{P}}(X_1; X_2, Y) + [I_{\tilde{P}'}(X_2; X_1, Y) - R_2]^+ \quad (129)$$

$$R_1 \leq \min_{(\tilde{P}_{X_1 X_2 Y}, \tilde{P}'_{X_1 X_2 Y}) \in \mathcal{T}_1^{(2,2)}(Q_1 \times Q_2 \times W, R_2)} I_{\tilde{P}}(X_1; X_2, Y) + [I_{\tilde{P}'}(X_2; X_1, Y) - R_2]^+, \quad (130)$$

where

$$\begin{aligned} \mathcal{T}_1^{(1,1)}(P_{X_1 X_2 Y}, R_2) &\triangleq \left\{ (\tilde{P}_{X_1 X_2 Y}, \tilde{P}'_{X_1 X_2 Y}) : \tilde{P}_{X_1 X_2 Y} \in \mathcal{S}_1(Q_1, P_{X_2 Y}), \right. \\ &\left. \tilde{P}'_{X_1 X_2 Y} \in \mathcal{S}'_1(Q_2, \tilde{P}_{X_1 Y}), I_{\tilde{P}'}(X_2; X_1, Y) \leq R_2, \mathbb{E}_{\tilde{P}}[\log q(X_1, X_2, Y)] \geq \underline{F}(P_{X_1 X_2 Y}, R_2) \right\} \end{aligned} \quad (131)$$

$$\begin{aligned} \mathcal{T}_1^{(1,2)}(P_{X_1 X_2 Y}, R_2) &\triangleq \left\{ (\tilde{P}_{X_1 X_2 Y}, \tilde{P}'_{X_1 X_2 Y}) : \tilde{P}_{X_1 X_2 Y} \in \mathcal{S}_1(Q_1, P_{X_2 Y}), \right. \\ &\left. \tilde{P}'_{X_1 X_2 Y} \in \mathcal{S}'_1(Q_2, \tilde{P}_{X_1 Y}), I_{\tilde{P}'}(X_2; X_1, Y) \leq R_2, \right. \\ &\left. \mathbb{E}_{\tilde{P}}[\log q(X_1, X_2, Y)] + R_2 - I_{\tilde{P}'}(X_2; X_1, Y) \geq \underline{F}(P_{X_1 X_2 Y}, R_2) \right\} \end{aligned} \quad (132)$$

$$\begin{aligned} \mathcal{T}_1^{(2,1)}(P_{X_1 X_2 Y}, R_2) &\triangleq \left\{ (\tilde{P}_{X_1 X_2 Y}, \tilde{P}'_{X_1 X_2 Y}) : \tilde{P}_{X_1 X_2 Y} \in \mathcal{S}_1(Q_1, P_{X_2 Y}), \right. \\ &\left. \tilde{P}'_{X_1 X_2 Y} \in \mathcal{S}'_1(Q_2, \tilde{P}_{X_1 Y}), \mathbb{E}_{\tilde{P}}[\log q(X_1, X_2, Y)] \geq \underline{F}(P_{X_1 X_2 Y}, R_2) \right\} \end{aligned} \quad (133)$$

$$\begin{aligned} \mathcal{T}_1^{(2,2)}(P_{X_1 X_2 Y}, R_2) &\triangleq \left\{ (\tilde{P}_{X_1 X_2 Y}, \tilde{P}'_{X_1 X_2 Y}) : \tilde{P}_{X_1 X_2 Y} \in \mathcal{S}_1(Q_1, P_{X_2 Y}), \right. \\ &\left. \tilde{P}'_{X_1 X_2 Y} \in \mathcal{S}'_1(Q_2, \tilde{P}_{X_1 Y}), \mathbb{E}_{\tilde{P}}[\log q(X_1, X_2, Y)] \geq \underline{F}(P_{X_1 X_2 Y}, R_2) \right\}. \end{aligned} \quad (134)$$

These are obtained from $\mathcal{T}^{(k)}$ ($k = 1, 2$) by keeping only one term in the definition of \bar{F}_k (see (66)–(67)), and by removing the constraint $I_{\tilde{P}'}(X_2; X_1, Y) \geq R_2$ when $k = 2$ in accordance with the discussion following (125).

The variable $\tilde{P}'_{X_1 X_2 Y}$ can be removed from both (127) and (129), since in each case the choice $\tilde{P}'_{X_1 X_2 Y}(x_1, x_2, y) = Q_2(x_2)\tilde{P}_{X_1 Y}(x_1, y)$ is feasible and yields $I_{\tilde{P}'}(X_2; X_1, Y) = 0$. It follows that (127) and (129) yield the same value, and we conclude that (18) can equivalently be expressed in terms of three conditions: (128), (130), and

$$R_1 \leq \min_{\tilde{P}_{X_1 X_2 Y} \in \mathcal{T}_1^{(1,1')} (Q_1 \times Q_2 \times W, R_2)} I_{\tilde{P}}(X_1; X_2, Y), \quad (135)$$

where the set

$$\mathcal{T}_1^{(1,1')} (P_{X_1 X_2 Y}, R_2) \triangleq \left\{ \tilde{P}_{X_1 X_2 Y} \in \mathcal{S}_1(Q_1, P_{X_2 Y}) : \mathbb{E}_{\tilde{P}}[\log q(X_1, X_2, Y)] \geq \underline{F}(P_{X_1 X_2 Y}, R_2) \right\} \quad (136)$$

is obtained by eliminating $\tilde{P}'_{X_1 X_2 Y}$ from either (131) or (133). These three conditions are all written as convex optimization problems, as desired.

Starting with (117)–(118), one can follow a (a simplified version of) the above arguments for the cognitive MAC to show that (30) holds if and only if

$$R_1 \leq \min_{(\tilde{P}_{X_1Y}, \tilde{P}'_{X_1X_2Y}) \in \mathcal{T}_{1c}^{(1)}(Q_{X_1X_2} \times W, R_2)} I_{\tilde{P}}(X_1; Y) \quad (137)$$

$$R_1 \leq \min_{(\tilde{P}_{X_1Y}, \tilde{P}'_{X_1X_2Y}) \in \mathcal{T}_{1c}^{(2')}(Q_{X_1X_2} \times W, R_2)} I_{\tilde{P}}(X_1; Y) + [I_{\tilde{P}'}(X_2; Y|X_1) - R_2]^+. \quad (138)$$

where

$$\mathcal{T}_{1c}^{(2')}(P_{X_1X_2Y}, R_2) \triangleq \left\{ (\tilde{P}_{X_1Y}, \tilde{P}'_{X_1X_2Y}) : \tilde{P}_{X_1Y} \in \mathcal{S}_{1c}(Q_{X_1}, P_Y), \right. \\ \left. \tilde{P}'_{X_1X_2Y} \in \mathcal{S}'_{1c}(Q_{X_1X_2}, \tilde{P}_{X_1Y}), \mathbb{E}_{\tilde{P}'}[\log q(X_1, X_2, Y)] \geq \underline{F}(P_{X_1X_2Y}, R_2) \right\}, \quad (139)$$

and where $\mathcal{T}_{1c}^{(1)}$, \mathcal{S}_{1c} and \mathcal{S}'_{1c} are defined in (113)–(116).

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